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The cubic Hecke algebra on at most 5 strands

Ivan Marin

Institut de Mathématiques de Jussieu, Université Paris 7, France

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In memory of Johann Gustav Hermes, who worked 10 years on completing the construction of the 65537-gon and on producing the corresponding beautiful artwork of drawings and numbers, nowadays known as 'Der Koffer' in Göttingen's library.

MSC: 20F36; 20C08

ABSTRACT

We prove that the quotient of the group algebra of the braid group on 5 strands by a generic cubic relation has finite rank. This was conjectured by Broué, Malle and Rouquier and has for consequence that this algebra is a flat deformation of the group algebra of the complex reflection group G_{32}, of order 155,520.

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1. Introduction

In 1957 H.S.M. Coxeter proved (see [8]) that the quotient of the braid group B_n on $n \geq 2$ strands by the relations $s_i^k = 1$, where $s_1, \dots, s_{n-1}$ denote the usual Artin generators, is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$. This means that, besides the obvious case $k = 2$, which leads to the symmetric group, and the case $n = 2$, there is only a finite number of such groups. They all turn out to be irreducible complex reflection groups, namely finite subgroups of $\mathrm{GL}_n(\mathbf{C})$ generated by endomorphisms which fix a hyperplane (so-called pseudo-reflections), and which leave no proper subspace invariant. In the usual classification of such objects, due to Shephard and Todd, they are nicknamed as G_4, G_8, G_{16} for $n = 3$ and $k = 3, 4, 5$, G_{25}, G_{32} for $n = 4, 5$ and $k = 3$.

In 1998, M. Broué, et al. conjectured (see [6]) that the group algebra of all complex reflection groups admit flat deformations similar to the Hecke algebra of a Weyl or Coxeter group. They actually introduced natural deformations of such group algebras, called them the (generic) Hecke algebra associated to such a group, and they conjectured that these were flat deformations, and in particular that they have finite rank. For the groups we are interested in, this conjecture actually amounts to saying that the quotients of the group algebra RB_n by the relations $s_i^k + a_{k-1}s_i^{k-1} + \cdots + a_1 s_i + a_0 = 0$, where $R = \mathbf{Z}[a_{k-1}, \dots, a_1, a_0, a_0^{-1}]$, is a flat deformation of the group algebra RW, where $W = B_n/s_i^k$ (note that we actually use a slightly smaller ring than the one used in [6] and [5]). This conjecture was proved in [5] for G_4 and G_{25} but not for the largest cubic case G_{32} (Satz 4.7 – the proof for G_{25} is however only sketched there); actually, a preliminary version of the conjecture (where Hecke algebras were not associated to an arbitrary complex reflection group but instead to a specific kind of group presentation), already covering the cases that we are considering here, dates back to 1993, and is stated in [5] (Vermutung 4.6).

Note that, outside its original framework, the validity of this conjecture is assumed in a number of papers about so-called Cherednik algebras and related topics.

E-mail address: marin@math.jussieu.fr.

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According to [6] (see the proof of theorem 4.24 there) only the following needs to be proved: that the algebra is spanned over R by $|W|$ elements. This is what we prove here.

Theorem 1.1. *The generic Hecke algebra associated to $W = G_{32}$ is spanned by $|W|$ elements, and is thus a free R-module of rank $|W|$ which becomes isomorphic to the group algebra of W after a suitable extension of scalars.*

More precisely, according to [11] corollary 7.2, a convenient extension of scalars would be $\mathbf{Q}(\zeta_3, (\zeta_3^{-r}u_r)^{\frac{1}{6}}, r = 0, 1, 2)$ where ζ_3 is a primitive 3rd root of 1 and $X^3 + a_2X^2 + a_1X + a_0 = (X - u_0)(X - u_1)(X - u_2)$ or, better, the algebraic extension of $\mathbf{Q}(\zeta_3)(u_0, u_1, u_2)$ generated by $\sqrt{u_0u_1}$, $\sqrt{u_0u_2}$, $\sqrt{u_1u_2}$ and $\sqrt[3]{u_0u_1u_2}$ (see [11] table 8.2 and proposition 5.1).

In the general setting of complex reflection groups, it is known that this conjecture is true

- for the general series (usually denoted $G(de, e, r)$) of complex reflection groups (by works Ariki–Koike [2] and Ariki [1]),
- for most of the exceptional groups of rank 2 by [5] and [14], which are numbered G_4 to G_{22}. More precisely, only the groups G_{17}, G_{18} and G_{19} have not been checked yet. In [9], a weak version of the conjecture is proved for all exceptional groups of rank 2.
- for G_{25} by [5], for the groups G_{26}, G_{27} by computer means ([14]).
- for the Coxeter groups.

The remaining cases are in rank 4 the groups G_{29} ([14] however checked that the algebra has the right dimension over the field of fractions), G_{31}, G_{32}, in rank 5 the group G_{33} and in rank 6 the group G_{34}. All but G_{32}, whose case we settled here, have all their pseudo-reflections of order 2.

In the case studied here, we actually prove more. Here and in the sequel we denote A_n the quotient of RB_n by the generic cubic relation $s_i^3 - as_i^2 - bs_i - c = 0$. The usual embedding $B_n \hookrightarrow B_{n+1}$ induces a natural morphism $A_n \to A_{n+1}$, hence an A_n-bimodule structure on A_{n+1}. For $n \le 4$, we give a decomposition of A_{n+1} as A_n-bimodule. This immediately provides an explicit R-basis of A_n for $n \le 5$, made of images of braids in B_n. Recall that the orders of G_4, G_{25} and G_{32} are 24, 648 and 155,520.

The following theorem is a recollection of the main results of this article: see in particular Theorems 3.2, 4.1, 6.21 and 6.26 as well as Corollary 5.12, and recall that the argument of [6] theorem 4.24 (which involves a transcendental monodromy construction) shows that proving that the Hecke algebra of type W is R-generated by $|W|$ elements ensures that this Hecke algebra is free as an R-module, with basis the given $|W|$ elements. Moreover, notice that, if we have an inclusion of parabolic subgroups $W_0 \subset W$ with corresponding Hecke algebras $H_0 \subset H$, knowing the conjecture for H_0 and that H is generated by $|W/W_0|$ elements as an H_0-module proves (1) the conjecture for H and (2) that H is free as an H_0-module, with basis these elements. Indeed, letting $N = |W/W_0|$ the assumption provides an H_0-module morphism $H_0^N \to H$; composing with $(R^{|W_0|})^N \simeq H_0^N$ this yields a surjective morphism $R^{|W|} \to H$ which is an isomorphism by the argument of [6]. This proves that the original morphism $H_0^N \to H$ has no kernel either, and so is an isomorphism.

Theorem 1.2.
- *Let $S_2 = \{1, s_1, s_1^{-1}\} \subset B_2$. One has $|S_2| = 3$ and S_2 provides an R-basis of A_2.*
- *Let $S_3 = S_2 \sqcup S_2s_2^{\pm}S_2 \sqcup S_2s_2^{-1}s_1s_2^{-1} \subset B_3$. One has $|S_3| = 24$ and S_3 provides an R-basis of A_3.*
- *A_4 is a free A_3-module of rank 27. A basis of this A_3-module is provided by elements of the braid group (including 1) which map to a system of representatives of G_{25}/G_4.*
- *A_4 is a free R-module of rank 648. A basis of this R-module is provided by elements of the braid group including 1 which map to all G_{25}.*
- *A_4 is a free $A_2 \otimes_R A_2 \simeq \langle s_1, s_3\rangle$-module of rank 72. A basis of this $\langle s_1, s_3\rangle$-module is provided by elements of the braid group including 1 which map to a system of representatives of $G_{25}/(\mathbf{Z}/3\mathbf{Z})^2$.*
- *A_5 is a free A_4-module of rank 240. A basis is provided by elements of the braid group including 1 which map to a system of representatives of G_{32}/G_{25}.*
- *A_5 is a free R-module of rank 155,520. A basis of this R-module is provided by elements of the braid group which include 1 and which map to all G_{32}.*

Corollary 1.3. *The natural map $A_n \to A_{n+1}$ is injective for $2 \le n \le 4$.*

We describe the plan of the proof. Our method is inductive. We find generators of A_{n+1} as an A_n-bimodule, and only then as an A_n-module. After some preliminaries in Section 2 we do the case of A_3 in Section 3. The structure of A_4 as an A_3-module is obtained in Section 4. Before considering A_5, we provide in Section 5 an alternative description of A_4, this time as a $\langle s_1, s_3\rangle$-module. In addition to providing an alternative proof of the conjecture for A_4, this is used in the decomposition of A_5 as an A_4-module. This decomposition is obtained in Section 6. We first obtain a decomposition of A_5 as an A_4-bimodule, and introduce a filtration of A_5 by simpler A_4-bimodules. The latest step of the filtration has original generators originating from the center of the braid group, and this turns out to be the crucial reason why this filtration terminates, thus proving that A_5 is an R-module of finite rank. For proving this crucial property one needs a lengthy calculation which is postponed in Section 7. We conclude the Section 6 and the proof of the main theorem by studying the structure as A_4-modules of the A_4-bimodules involved there.

Remark 1.4. A detailed version of this paper, with more computations detailed, can be found on the arxiv. For publication purposes, we skip here the details for quite a few computations. In particular, we assert without proof the equalities between words in s_1, s_2, s_3, s_4 when they are (sometimes not so easy) identities inside the braid group. By using normal forms for elements in the braid group, the verification of such identities can be easily automatized.

1.1. Perspectives

It seems likely that our methods can be used to attack the conjecture for other complex reflection groups of higher rank. One indeed has the following standard inclusions of parabolic subgroups (except for the dotted line, which is not a parabolic inclusion). The number associated to the inclusion is the number of double classes. Note again that the groups of rank at least 3 for which the conjecture remains open have all their reflections of order 2.

$$\begin{tikzcd}
G(3,3,2) \arrow[r, hook, "4"'] & G(3,3,3) \arrow[r, hook, "4"'] & G(3,3,4) \arrow[r, hook, "6"'] & G_{33} \arrow[r, hook, "13"'] & G_{34}
\end{tikzcd}$$

$$\begin{tikzcd}
G(4,4,2) \arrow[r, hook, "5"'] & G(4,4,3) \arrow[r, hook, "9"'] & G_{29} \arrow[d, dotted, hook, "2"] \\
 & G(2,1,3) \arrow[ur, hook, "16"'] & G_{31}
\end{tikzcd}$$

For instance, 8 of the 9 double classes of $W = G_{29} = \langle g_1, g_2, g_3, g_4\rangle$ with respect to $W_0 = G(4,4,3) = \langle g_2, g_3, g_4\rangle$ have for representatives $g_1^{\varepsilon}z$ for $z \in Z(W)$ and $\varepsilon \in \{0,1\}$. If we had a practical knowledge of the braid groups of type G_{29} and $G(4,4,3)$ of the same level than the one we have for the usual braid group, the methods used here would then probably yield a proof of the conjecture for G_{29} in the same way we managed to get one for G_{32}, as this kind of phenomenon (that the most complicated double classes are mainly represented by central elements) is crucial in our proof. Similarly, if $G_{34} = \langle s_1, \dots, s_6\rangle$ with $G_{33} = \langle s_1, \dots, s_5\rangle$, one can check that 12 of the 13 double classes have for representative a term of the form zs_6^{ε} for $\varepsilon \in \{0,1\}$ and z a central element of G_{34}.

Another natural question is whether similar deformations exist for a higher number of strands. Indeed, although it is known that the groups $\Gamma_n = B_n/s_i^3$ are infinite for $n \geq 6$, it was proved in [3] (see also [7]) that $\Gamma_n^{(1)} = \Gamma_n/z_5^2$ and $\Gamma_n^{(2)} = \Gamma_n/z_5^3$ are finite for arbitrary $n \geq 5$, and are related to symplectic group over $\mathbf{F}_3$ and to unitary groups over $\mathbf{F}_2$, respectively. Here z_5 denotes the image of the generator $(s_1s_2s_3s_4)^5$ of the braid group on 5 strands into Γ_n, $n \geq 5$, which has order 6 in Γ_5. It is thus tempting to look for deformations of the group algebras of $\Gamma_n^{(1)}$ and $\Gamma_n^{(2)}$ for arbitrary n that would be quotients of the group algebra of the braid group by a generic cubic relations *and* other relations probably involving z_5.

1.2. Applications

We mention the following consequences. A first one concerns the study of linear representations of the (usual) braid groups. A consequence of the proof in [5] for the cases G_4, G_8 and G_{16} was a classification of the linear representations of the braid group B_3 in which the image of s_1 (and thus of all s_i) is killed by a polynomial of degree at most 5 : indeed, such a representation has to factorize through the corresponding Hecke algebra. This proves that such representations have a very rigid structure, a result rediscovered in [15]. A similar consequence of this new result is a classification of the linear representations of the braid group B_n for n at most 5 in which the image of s_1 is killed by a cubic polynomial.

A second one is about the cubic invariants of knots and links. The algebras connected to cubic invariants, including the Kauffman polynomial and the Links–Gould polynomial (which is studied in [13]), are quotients of A_n. Our result gives the structure of A_5; in order to prove it, we actually establish its decomposition as an A_4-bimodule, which may be useful in order to understand the possible Markov traces factorizing through A_n.

Specifically, in [7], we used the representation theory of the group G_{32} to prove that an algebra $K_n(\gamma)$ introduced by Funar in [10] for studying knot invariants collapsed for large n over a field of characteristic distinct from 2, and in characteristic 0 for $n \geq 5$. An immediate consequence of the present result is that our argument in characteristic 0 applies verbatim to prove that the deformation $K_n(\alpha, \beta)$ of $K_n(\gamma)$ introduced by Bellingeri and Funar in [4] also collapses for $n \geq 5$. We provide the details below.

Theorem 1.5. *The generic algebra $K_n(\alpha, \beta)$ introduced in [4] is zero for $n \geq 5$.*

Proof. We use the notations of [4]. Let $\mathbf{k}$ be an algebraically closed extension of $\mathbf{Q}(\alpha, \beta)$. The $\mathbf{Z}[\alpha, \beta]$-algebra $K_n(\alpha, \beta)$ is defined as the quotient of the group algebra $\mathbf{Z}[\alpha, \beta]B_n$ by the two-sided ideal generated by the elements $s_i^3 - \alpha s_i^2 - \beta s_i - 1$ and another element $q \in \mathbf{Z}[\alpha, \beta]B_3 \subset \mathbf{Z}[\alpha, \beta]B_n$. We let $\varphi : \mathbf{Z}[a, b, c, c^{-1}] \to \mathbf{Z}[\alpha, \beta]$ be the specialization $a \mapsto \alpha$, $b \mapsto \beta$, $c \mapsto 1$, and let A_n^0 denote $A_n \otimes_{\varphi} \mathbf{Z}[\alpha, \beta]$. Obviously $K_n(\alpha, \beta)$ is a quotient of A_n^0, more precisely the quotient of A_n^0 by the two-sided ideal generated by (the canonical image of) q. Let $\mathbf{k}$ denote an algebraically closed extension of $\mathbf{Q}(\alpha, \beta)$. We have $A_3^0 \otimes_{\mathbf{Z}[\alpha,\beta]} \mathbf{k} \simeq \mathbf{k}G_4 \simeq \mathbf{k}^3 \oplus Mat_2(\mathbf{k})^3 \oplus Mat_3(\mathbf{k})$, and the ideal generated by q is by definition the factor $\mathbf{k}^3$ in this decomposition (see remark 1.3 in [4]). As a consequence, the $\mathbf{k}$-algebra $\mathbf{k}K_5(\alpha, \beta)$ is the quotient of the semisimple algebra $\mathbf{k}A_5^0 \simeq \mathbf{k}G_{32}$ by the following two-sided ideal : make the direct sum of all the direct factors $Mat_N(\mathbf{k})$ whose corresponding irreducible representations have at least one 1-dimensional component in their restriction to $\mathbf{k}A_3^0$. Now, to the expense of possibly enlarging $\mathbf{k}$, the isomorphisms between the algebras A_n^0 and the corresponding group algebras can be chosen in such a way that the following diagram commutes (e.g. by theorem 2.9 of [12] — see also remark 2.11 there).

$$\begin{tikzcd}
\mathbf{k}A_3^0 \arrow[r] \arrow[d] & \mathbf{k}A_4^0 \arrow[r] \arrow[d] & \mathbf{k}A_5^0 \arrow[d] \\
\mathbf{k}G_4 \arrow[r] & \mathbf{k}G_{25} \arrow[r] & \mathbf{k}G_{32}
\end{tikzcd}$$

As in [7], the induction table between the (ordinary) characters of G_4 of G_{32} then shows that *all* direct factors $Mat_N(\mathbf{k})$ satisfy this property, and thus the two-sided ideal is all A_5^0. It follows that $K_5(\alpha, \beta) = 0$, whence $K_n(\alpha, \beta) = 0$ for $n \geq 5$, as $K_n(\alpha, \beta)$ is generated by conjugates of the image of $K_5(\alpha, \beta)$. $\square$

2. Preliminaries and notations

We let $R = \mathbf{Z}[a, b, c, c^{-1}]$ and let B_n denote the braid group on n strands, generated by the braids $s_1, \dots, s_{n-1}$ with relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for $|j - i| \geq 2$. The cubic Hecke algebra A_n for $n \geq 2$ is the quotient of the group algebra RB_n by the relations $s_i^3 = as_i^2 + bs_i + c$. We identify s_i to their images in A_n. Notice that, since c is invertible in R, s_i is still invertible, and we have the equivalent relations $s_i^2 = as_i + b + cs_i^{-1}$, etc.

The group algebra RB_n admits the automorphism $s_i \mapsto s_{n-i}$, which induces an automorphism of A_n, as a R-algebra. The automorphism $s_i \mapsto s_i^{-1}$ of B_n induces an automorphism Φ of A_n as a $\mathbf{Z}$-algebra, defined by $s_i \mapsto s_i^{-1}$, $a \mapsto -bc^{-1}$, $b \mapsto -ac^{-1}$, $c \mapsto c^{-1}$, and similarly the skew-automorphism Ψ of B_n defined by $s_i \mapsto s_i^{-1}$ induces a skew-automorphism of A_n as a $\mathbf{Z}$-algebra.

In the sequel we will denote u_i the R-subalgebra of A_n generated by s_i (or equivalently by s_i^{-1}).

The following equalities hold in the braid group, and thus also in A_n. We state them as a lemma because of their importance in the sequel. Notice that they transform an element of the form $s_{i+1}^{\pm} s_i^{\epsilon} s_{i+1}^{\mp}$ into an element of $u_i u_{i+1} u_i$.

Lemma 2.1. *For $\alpha \in \{-1, 1\}$, we have $s_{i+1}^{\alpha} s_i^{\alpha} s_{i+1}^{-\alpha} = s_i^{-\alpha} s_{i+1}^{\alpha} s_i^{\alpha}$ and $s_{i+1}^{\alpha} s_i^{-\alpha} s_{i+1}^{-\alpha} = s_i^{-\alpha} s_{i+1}^{-\alpha} s_i^{\alpha}$.*

Lemma 2.2. *For all $\epsilon \in \{0, 1, -1\}$, $s_{i+1}^{\pm} s_i^{\epsilon} s_{i+1}^{\mp} \in u_i u_{i+1} u_i$, $s_{i+1}^{\pm} s_i^{\pm} s_{i+1}^{\epsilon} \in u_i u_{i+1} u_i$, and $s_{i+1}^{\epsilon} s_i^{\pm} s_{i+1}^{\pm} \in u_i u_{i+1} u_i$.*

Proof. The first item is a direct consequence of Lemma 2.1, and the latter two items are consequences of the braid relations $s_i^{\pm} s_{i+1}^{\pm} s_i^{\pm} = s_i^{\pm} s_{i+1}^{\pm} s_i^{\pm}$. $\square$

Lemma 2.3. (1) *For all $x \in u_i$, $(s_{i+1}^{-1} s_i s_{i+1}^{-1})x \in x(s_{i+1}^{-1} s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i$.*
(2) *For all $x \in u_i$, $(s_{i+1} s_i^{-1} s_{i+1})x \in x(s_{i+1} s_i^{-1} s_{i+1}) + u_i u_{i+1} u_i$.*

Proof. (2) is a consequence of (1) up to applying an automorphism of A_n, so we restrict ourselves to proving (1). Since u_i is generated as a R-algebra by s_i^{-1}, we only need to prove $(s_{i+1}^{-1} s_i s_{i+1}^{-1}) s_i^{-1} \in s_i^{-1}(s_{i+1}^{-1} s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i$. We use $s_i = cs_i^{-2} + bs_i^{-1} + a$, $s_i^{-2} = c^{-1}s_i - ac^{-1} - bc^{-1}s_i^{-1}$ and the braid relations, and get

$$\begin{aligned}
(s_{i+1}^{-1} s_i s_{i+1}^{-1}) s_i^{-1} &= s_{i+1}^{-1} s_i s_{i+1}^{-1} s_i^{-1} \\
&= s_{i+1}^{-1}(cs_i^{-2} + bs_i^{-1} + a) s_{i+1}^{-1} s_i^{-1} \\
&= cs_{i+1}^{-1} s_i^{-2} s_{i+1}^{-1} s_i^{-1} + bs_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1} s_i^{-1} + as_{i+1}^{-1} s_{i+1}^{-1} s_i^{-1} \\
&= cs_{i+1}^{-1} s_i^{-2} s_{i+1}^{-1} s_i^{-1} + bs_i^{-1} s_{i+1}^{-1} s_i^{-1} s_i^{-1} + as_{i+1}^{-1} s_{i+1}^{-1} s_i^{-1} \\
&\in cs_{i+1}^{-1} s_i^{-1}(s_i^{-1} s_{i+1}^{-1} s_i^{-1}) + u_i u_{i+1} u_i \\
&\in c(s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1}) s_i^{-1} s_{i+1}^{-1} + u_i u_{i+1} u_i \\
&\in cs_i^{-1} s_{i+1}^{-1} s_i^{-2} s_{i+1}^{-1} + u_i u_{i+1} u_i \\
&\in cs_i^{-1} s_{i+1}^{-1}(c^{-1}s_i - ac^{-1} - bc^{-1}s_i^{-1}) s_{i+1}^{-1} + u_i u_{i+1} u_i \\
&\in s_i^{-1} s_{i+1}^{-1}(s_i - a - bs_i^{-1}) s_{i+1}^{-1} + u_i u_{i+1} u_i \\
&\in s_i^{-1} s_{i+1}^{-1} s_i s_{i+1}^{-1} - as_i^{-1} s_{i+1}^{-1} s_{i+1}^{-1} - bs_i^{-1}(s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1}) + u_i u_{i+1} u_i \\
&\in s_i^{-1} s_{i+1}^{-1} s_i s_{i+1}^{-1} - as_i^{-1} s_{i+1}^{-1} s_{i+1}^{-1} - bs_i^{-1} s_i^{-1} s_{i+1}^{-1} s_i^{-1} + u_i u_{i+1} u_i \\
&\in s_i^{-1}(s_{i+1}^{-1} s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i. \quad \square
\end{aligned}$$

Lemma 2.4. $s_{i+1}^{-1} s_i s_{i+1}^{-1} \in c^{-1}(s_{i+1} s_i^{-1} s_{i+1}) s_i + u_i u_{i+1} u_i$.

Proof. We have $(s_{i+1} s_i^{-1} s_{i+1}) s_i = s_{i+1}(s_i^{-1} s_{i+1} s_i) = s_{i+1} s_{i+1} s_i s_{i+1}^{-1} = s_{i+1}^2 s_i s_{i+1}^{-1}$ by Lemma 2.1. Since $s_{i+1}^2 = as_{i+1} + b + cs_{i+1}^{-1}$ we get $(s_{i+1} s_i^{-1} s_{i+1}) s_i = (as_{i+1} + b + cs_{i+1}^{-1}) s_i s_{i+1}^{-1} = as_{i+1} s_i s_{i+1}^{-1} + bs_i s_{i+1}^{-1} + cs_{i+1}^{-1} s_i s_{i+1}^{-1} \in cs_{i+1}^{-1} s_i s_{i+1}^{-1} + u_i u_{i+1} u_i$ since $s_{i+1} s_i s_{i+1}^{-1} \in u_i u_{i+1} u_i$ by Lemma 2.1. $\square$

3. The algebra A_3

We identify A_2 with its image in A_3 under $s_i \mapsto s_i$, that is with the subalgebra of A_3 generated by s_1. Lemma 2.1 provides the equalities $s_2s_1s_2^{-1} = s_1^{-1}s_2s_1$, $s_2s_1^{-1}s_2^{-1} = s_1^{-1}s_2^{-1}s_1$, $s_2^{-1}s_1s_2 = s_1s_2s_1^{-1}$, $s_2^{-1}s_1^{-1}s_2 = s_1s_2^{-1}s_1^{-1}$.

Lemma 3.1. *$s_2^{-1}s_1s_2^{-1}A_2 \subset A_2s_2^{-1}s_1s_2^{-1} + u_2u_1u_2$ and $s_2s_1^{-1}s_2A_2 \subset A_2s_2s_1^{-1}s_2 + u_2u_1u_2$.*

Proof. Straightforward consequences of Lemma 2.3 □

Theorem 3.2. (1) $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + A_2s_2^{-1}s_1s_2^{-1}A_2$
(2) $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + A_2s_2s_1^{-1}s_2A_2$
(3) $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + A_2s_2s_1^{-1}s_2 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + s_2s_1^{-1}s_2A_2$
(4) $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + A_2s_2^{-1}s_1s_2^{-1} = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + s_2^{-1}s_1s_2^{-1}A_2$.

Proof. Up to applying Φ, (2) is a consequence of (1). Then (3) and (4) are consequences of (1) and (2) by the above lemma. We now prove (1), and let U denote its RHS. It is clearly an A_2-submodule of A_3 which contains 1, so we only need to prove $s_2U \subset U$. Note that, clearly, $u_1u_2u_1 \subset U$. We first prove $u_2u_1u_2 \subset U$. Since we know $u_1u_2 \subset U$, $u_2u_1 \subset U$, this means that $w = s_2^{\alpha}s_1^{\beta}s_2^{\gamma} \in U$ for all $\alpha, \beta, \gamma \in \{-1, 1\}$. If α and β have opposite signs this element belongs to $u_1u_2u_1 \subset U$ by Lemma 2.1, so we can assume $\alpha = \beta$. If $\alpha = \beta = \gamma$, then the braid relations imply $w \in u_2u_1u_2 \subset U$. Thus only remains $w \in \{s_2^{-1}s_1s_2^{-1}, s_2s_1^{-1}s_2\}$. Clearly $s_2^{-1}s_1s_2^{-1} \in U$, and $s_2s_1^{-1}s_2 \in c(s_2^{-1}s_1s_2^{-1})s_1^{-1} + u_1u_2u_1 \subset u_1Uu_1 = U$ by Lemma 2.4. We thus proved $u_2u_1u_2 \subset U$. We now prove $s_2U \subset U$. Clearly $s_2(A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2) \subset u_2u_1u_2u_1 \subset Uu_1 \subset U$, so we need to prove $s_2u_1s_2^{-1}s_1s_2^{-1} \subset U$. But $s_2u_1s_2^{-1} \subset u_1u_2u_1$ by Lemma 2.1 hence $s_2u_1s_2^{-1}s_1s_2^{-1} \subset u_1u_2u_1u_2 \subset u_1U \subset U$. This proves the claim. □

Corollary 3.3. *We have $A_3 = u_1u_2u_1u_2 = u_2u_1u_2u_1$. Moreover,*

$$\begin{aligned} A_3 &= u_1u_2u_1 + u_2u_1u_2 + Rs_1^{-1}s_2s_1^{-1}s_2 = u_1u_2u_1 + u_2u_1u_2 + Rs_2^{-1}s_1s_2^{-1}s_1 \\ &= u_1u_2u_1 + u_2u_1u_2 + Rs_1s_2^{-1}s_1s_2 = u_1u_2u_1 + u_2u_1u_2 + Rs_2s_1^{-1}s_2s_1 \end{aligned}$$

Corollary 3.4. *Let $n \geq 3$. For all $1 \leq i, j \leq n-1$, we have in A_n the equality $u_iu_ju_iu_j = u_ju_iu_ju_i$.*

This theorem implies that A_3 is a free R-module of finite rank, consequently that $A_3 \subset A_3 \otimes_R K \simeq Mat_3(K) \oplus Mat_2(K)^3 \oplus K^3$ where K is a sufficiently large extension of the quotient field of R, and the isomorphism is explicitly given by the matrix models of the irreducible representations of A_3. From this it is simply a linear algebra matter to check equalities in A_3, or to express a given element in a given basis. In order to preserve the computer-free spirit of this paper, we however sketch a proof of the useful identities below.

Lemma 3.5.

$$\begin{aligned} s_1s_2^{-1}s_1s_2^{-1} &= s_2^{-1}s_1s_2^{-1}s_1 + \frac{a}{c}s_1s_2 - \frac{a}{c}s_2s_1 - \frac{ab}{c}s_1s_2^{-1} + \frac{ab}{c}s_2^{-1}s_1 + bs_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1} \\ s_2s_1^{-1}s_2s_1^{-1} &= s_2^{-1}s_1s_2^{-1}s_1 + a(s_1^{-1}s_2s_1^{-1} - s_2^{-1}s_1s_2^{-1}) - \frac{ab}{c}s_1s_2^{-1} + \frac{ab}{c}s_1^{-1}s_2 + \frac{b}{c}s_1s_2^{-1}s_1 - \frac{b}{c}s_2s_1^{-1}s_2 \\ s_1^{-1}s_2s_1^{-1}s_2 &= s_2^{-1}s_1s_2^{-1}s_1 + \frac{a}{c}s_1s_2 - as_2^{-1}s_1s_2^{-1} - \frac{a}{c}s_2s_1 + as_1^{-1}s_2s_1^{-1} \\ &\quad - \frac{ab}{c}s_1s_2^{-1} + \frac{b}{c}s_1s_2^{-1}s_1 + \frac{ab}{c}s_2s_1^{-1} - \frac{b}{c}s_2s_1^{-1}s_2 + bs_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1} \end{aligned}$$

Proof. We want to show the equality $s_1s_2^{-1}s_1s_2^{-1} = s_2^{-1}s_1s_2^{-1}s_1 + \frac{a}{c}s_1s_2 - \frac{a}{c}s_2s_1 - \frac{ab}{c}s_1s_2^{-1} + \frac{ab}{c}s_2^{-1}s_1 + bs_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1}$. The other identities are elementary consequences of these ones, through the usual (skew) automorphisms, and the conjugation by $s_1s_2s_1$ which exchanges s_1 and s_2. Multiplying this equality on the left by s_2, it is equivalent to $s_2s_1s_2^{-1}s_1s_2^{-1} = s_1s_2^{-1}s_1 + \frac{a}{c}s_2s_1s_2 - \frac{a}{c}s_2^2s_1 - \frac{ab}{c}s_2s_1s_2^{-1} + \frac{ab}{c}s_1 + bs_1^{-1} - bs_2s_1^{-1}s_2^{-1}$. We have

$$\begin{aligned} (s_2s_1s_2^{-1})s_1s_2^{-1} &= s_1^{-1}s_2(s_1^2)s_2^{-1} = a(s_1^{-1}s_2s_1)s_2^{-1} + bs_1^{-1} + cs_1^{-1}(s_2s_1^{-1}s_2^{-1}) \\ &= as_2s_1(s_2^{-2}) + bs_1^{-1} + cs_1^{-2}s_2^{-1}s_1 \\ &= \tfrac{a}{c}s_2s_1s_2 - \tfrac{ab}{c}s_2s_1s_2^{-1} - \tfrac{a^2}{c}s_2s_1 + bs_1^{-1} + s_1s_2^{-1}s_1 - bs_1^{-1}s_2^{-1}s_1 - as_2^{-1}s_1 \end{aligned}$$

Comparing this expression with the RHS of the desired identity, we conclude that it is equivalent to $-\frac{a^2}{c}s_2s_1 - as_2^{-1}s_1 = \frac{-a}{c}s_2^2s_1 + \frac{ab}{c}s_1$ which, after multiplication on the right by cs_1^{-1}, is equivalent to $a(s_2^2 - as_2 - cs_2^{-1} - b) = 0$, which holds true. This concludes the proof. □

As a immediate consequence, we get

Lemma 3.6.

$$\begin{aligned} s_1s_2^{-1}s_1s_2^{-1} - s_2^{-1}s_1s_2^{-1}s_1 &= \tfrac{a}{c}s_1s_2 - \tfrac{a}{c}s_2s_1 + -\tfrac{ab}{c}s_1s_2^{-1} + \tfrac{ab}{c}s_2^{-1}s_1 + bs_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1} \\ s_2s_1^{-1}s_2s_1^{-1} - s_1^{-1}s_2s_1^{-1}s_2 &= \tfrac{ab}{c}s_1^{-1}s_2 - \tfrac{a}{c}s_1s_2 + \tfrac{a}{c}s_2s_1 - \tfrac{ab}{c}s_2s_1^{-1} - bs_2^{-1}s_1^{-1} + bs_1^{-1}s_2^{-1} \end{aligned}$$

A few additional computations lead to the following identity.

Lemma 3.7.

$$\begin{aligned} s_2^{-1}s_1s_2^{-1}s_1s_2^{-1}s_1 = {} & \tfrac{-(c+ab)a}{c^2}s_1 + \tfrac{a}{c}s_1s_2 + \tfrac{a}{c}s_1^{-1}s_2s_1 - \tfrac{ab}{c}s_2^{-1}s_1s_2^{-1} + \tfrac{-ab}{c}s_1^{-1} + \tfrac{ab}{c^2}s_2s_1 \\ & + s_1^{-1}s_2s_1^{-1} - \tfrac{b}{c}s_2^{-1}s_1s_2^{-1}s_1 - \tfrac{ab^2}{c^2}s_2^{-1}s_1 + \tfrac{b}{c}s_1^{-1}s_2 \\ & - \tfrac{a}{c}s_1s_2^{-1}s_1 + \tfrac{b}{c}s_2s_1^{-1} - \tfrac{b^2}{c}s_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1}s_1^{-1} \end{aligned}$$

4. The algebra A_4 as an A_3 (bi)module

We identify A_3 with its image in A_4, and denote $sh(A_3)$ the R-subalgebra of A_4 generated by s_2, s_3, s_4. It is the image of A_3 under the 'shift' morphism $s_i \mapsto s_{i+1}$. The goal of this section is to prove the following theorem.

Theorem 4.1. (1) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}A_3 + A_3s_3s_2^{-1}s_1s_2^{-1}s_3A_3$
(2) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + A_3s_3s_2^{-1}s_1s_2^{-1}s_3$
(3) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}A_3 + s_3s_2^{-1}s_1s_2^{-1}s_3A_3$.

We denote U the right-hand side of (1). We notice that $sh(A_3) \subset U$, because of Theorem 3.2(2). Also notice that $\Psi(U) = U$ and $\Phi(U) = U$ because of Lemma 2.4.

Lemma 4.2. $u_3A_3u_3 \subset U$.

Proof. By Theorem 3.2 we have $A_3 = u_1u_2u_1 + u_1s_2^{-1}s_1s_2^{-1}$ hence $u_3A_3u_3 \subset u_3u_1u_2u_1u_3 + u_3u_1s_2^{-1}s_1s_2^{-1}u_3$. But $u_3u_1u_2u_1u_3 = u_1u_3u_2u_3u_1 \subset u_1sh(A_3)u_1 \subset u_1Uu_1 \subset U$, and $u_3u_1s_2^{-1}s_1s_2^{-1}u_3 = u_1u_3s_2^{-1}s_1s_2^{-1}u_3$ so we need to prove $s_3^{\alpha}s_2^{-1}s_1s_2^{-1}s_3^{\beta} \in U$ for $\alpha, \beta \in \{-1, 1\}$. The case $(\alpha, \beta) = (1, 1)$ is clear by definition of U. When $(\alpha, \beta) = (-1, -1)$, we have $s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1} \in c^{-1}s_3^{-1}s_2s_1^{-1}s_2s_1s_3^{-1} + s_3^{-1}u_1u_2u_1s_3^{-1}$ that is $s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1} \in s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_1 + u_1s_3^{-1}u_2s_3^{-1}u_1 \subset U + u_1sh(A_3)u_1 \subset U$. When $(\alpha, \beta) = (1, -1)$ we get $s_3s_2^{-1}s_1s_2^{-1}s_3^{-1} = s_3s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2^{-1})s_2 = s_3s_2^{-1}s_1s_3^{-1}s_2^{-1}s_3^{-1}s_2 = (s_3s_2^{-1}s_3^{-1})s_1s_2^{-1}s_3^{-1}s_2 = s_2^{-1}s_3^{-1}(s_2s_1s_2^{-1})s_3^{-1}s_2 \in s_2^{-1}s_3^{-1}u_1u_2u_1s_3^{-1}s_2 \subset s_2^{-1}u_1s_3^{-1}u_2s_3^{-1}u_1s_2 \subset A_3sh(A_3)A_3 \subset U$. The case $(-1, 1)$ is similar. □

Lemma 4.3. $u_3A_3u_3A_3u_3 \subset A_3u_3A_3u_3A_3$.

Proof. For $x \in A_3$, we say that x has at most p factors if it belongs to as $u_{\sigma(1)} \dots u_{\sigma(p)}$ for some $\sigma : [1, p] \to \{1, 2\}$. By Theorem 3.2 the minimal number of factors for such an x is at most 4. We let $x, y \in A_3$, with minimal number of factors p and q, and prove that $u_3xu_3yu_3 \subset A_3u_3A_3u_3A_3$ by induction on (p, q) in lexicographic order. Note that, since $\Psi(U) = U$, we may assume $p \geq q$. Moreover, since $A_3 = u_1u_2u_1u_2$ and $u_3(u_1u_2u_1u_2)u_3yu_3 = u_1u_3u_2u_1u_2u_3yu_3$, we can assume $p \leq 3$ (hence $q \leq 3$).

The case $q = 0$ is trivial. If $x \in u_1u_{\sigma(2)} \dots u_{\sigma(p)}$, we have $u_3xu_3yu_3 \in u_3u_1u_{\sigma(2)} \dots u_{\sigma(p)}u_3yu_3 = u_1u_3u_{\sigma(2)} \dots u_{\sigma(p)}u_3yu_3$ and we are reduced to the case $(p-1, q)$. Similarly, if $y \in u_{\tau(1)} \dots u_{\tau(q-1)}u_1$, we are reduced to $(p, q-1)$. As a consequence, the only non-trivial case for $p \leq 1$ is $u_3u_2u_3u_2u_3 \subset sh(A_3) \subset A_3u_3A_3u_3A_3$ because of Theorem 3.2.

We consider the case $(p, q) = (2, 1)$. The only nontrivial case is $u_3u_2u_1u_3u_2u_3$. We need to prove $s_3^{\alpha}u_2u_1s_3^{\beta}u_2s_3^{\gamma} \subset A_3u_3A_3u_3A_3$ for all $\alpha, \beta, \gamma \in \{-1, 1\}$. Because $s_3^{\alpha}u_2u_1(s_3^{\beta}u_2s_3^{\gamma}) = (s_3^{\alpha}u_2s_3^{\beta})u_1u_2s_3^{\gamma}$ this is clear by Lemma 2.1 unless α, β and γ are all the same. We thus need to prove $s_3^{\alpha}s_2^{\beta}u_1s_3^{\alpha}s_2^{\gamma}s_3^{\alpha} \subset A_3u_3A_3u_3A_3$ for all $\beta, \gamma \in \{-1, 1\}$. Since $s_3^{\alpha}s_2^{\alpha}s_3^{\alpha} = s_2^{\alpha}s_3^{\alpha}s_2^{\alpha}$ we can assume $\beta = -\alpha$ and $\gamma = -\alpha$ and consider $s_3^{\alpha}s_2^{-\alpha}u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}$. By Lemma 2.4 we have $s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha} \in s_3^{-\alpha}s_2^{\alpha}s_3^{-\alpha}u_2 + u_2u_3u_2$ hence $s_3^{\alpha}s_2^{-\alpha}u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha} \subset s_3^{\alpha}s_2^{-\alpha}u_1s_3^{-\alpha}s_2^{\alpha}s_3^{-\alpha}u_2 + s_3^{\alpha}s_2^{-\alpha}u_1u_2u_3u_2 \subset s_3^{\alpha}s_2^{-\alpha}u_1s_3^{-\alpha}s_2^{\alpha}s_3^{-\alpha}u_2 + u_3A_3u_3A_3$ and we already noticed

$$s_3^{\alpha}s_2^{-\alpha}u_1s_3^{-\alpha}s_2^{\alpha}s_3^{-\alpha}u_2 = (s_3^{\alpha}s_2^{-\alpha}s_3^{-\alpha})u_1s_2^{\alpha}s_3^{-\alpha}u_2 \subset u_2u_3u_2u_1u_2u_3u_2 \subset A_3u_3A_3u_3A_3.$$

All cases for $(p, q) = (2, 2)$ can be easily reduced to smaller values by commutation relations. The only a priori irreducible case for $(p, q) = (3, 1)$ is $u_3u_2u_1u_2u_3u_2u_3$. Since $u_2u_3u_2u_3 \subset u_2u_3u_2 + u_3u_2u_3$ by Theorem 3.2, we are reduced to case (2, 1).

For the case $(p, q) = (3, 2)$, we can use a similar argument: the only nontrivial case is $u_3u_2u_1u_2u_3u_1u_2u_3 = u_3u_2u_1u_2u_1u_3u_2u_3$ and $u_2u_1u_2u_1 \subset u_2u_1u_2 + u_1u_2u_1$, and we are reduced to smaller cases.

The only remaining case is thus $(p, q) = (3, 3)$. Since $x \in A_3 = u_1u_2u_1 + u_1s_2^{-1}s_1s_2^{-1}$ and $y \in A_3 = u_1u_2u_1 + s_2^{-1}s_1s_2^{-1}u_1$ we are reduced to considering $s_3^{\alpha}s_2^{-1}s_1s_2^{-1}s_3^{\beta}s_2^{-1}s_1s_2^{-1}s_3^{\gamma}$ for $\alpha, \beta, \gamma \in \{-1, 1\}$. Up to applying Φ if necessary, we can assume $\beta = -1$. Then $s_3^{\alpha}s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2^{-1}s_1s_2^{-1}s_3^{\gamma} = s_3^{\alpha}s_2^{-1}s_1s_3^{-1}s_2^{-1}s_3^{-1}s_1s_2^{-1}s_3^{\gamma} = (s_3^{\alpha}s_2^{-1}s_3^{-1})s_1s_2^{-1}s_1(s_3^{-1}s_2^{-1}s_3^{\gamma}) \subset u_2u_3u_2A_3u_2u_3u_2 \subset A_3u_3A_3u_3A_3$ by Lemma 2.2, and this concludes the proof. □

We let $U_0 = A_3u_3A_3 + A_3s_3s_2^{-1}s_3A_3 = A_3u_3A_3 + A_3s_3^{-1}s_2s_3^{-1}A_3 = A_3sh(A_3)A_3 \subset U$.

Lemma 4.4. (1) $s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}A_3 \subset A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + U_0$
(2) $s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}A_2 \subset A_2s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + U_0$
(3) $s_3s_2^{-1}s_1s_2^{-1}s_3A_3 \subset A_3s_3s_2^{-1}s_1s_2^{-1}s_3 + U_0$
(4) $s_3s_2^{-1}s_1s_2^{-1}s_3A_2 \subset A_2s_3s_2^{-1}s_1s_2^{-1}s_3 + U_0$.

Statements (3) and (4) are consequences of (1) and (2) by application of Φ, and (1) and (2) are immediate consequences of the more detailed lemma below.

Lemma 4.5. (1) *For all* $x \in A_2$, $s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}x \in xs_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + U_0$.
(2) $(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})s_2^{-1} \in s_1^{-1}s_2^{-1}s_1(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}) + U_0$
(3) *For all* $x \in A_3$, $(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})x \in x^{s_1}(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}) + U_0$ *(where* $x^{s_1} = s_1^{-1}xs_1$*).*

Proof. We first prove (1). We have

$$\begin{aligned}(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})s_1^{-1} &= s_3^{-1}(s_2s_1^{-1}s_2)s_1^{-1}s_3^{-1} \\ &\in s_3^{-1}s_1^{-1}(s_2s_1^{-1}s_2)s_3^{-1} + s_3^{-1}u_1u_2u_1s_3^{-1} \\ &\subset s_3^{-1}s_1^{-1}(s_2s_1^{-1}s_2)s_3^{-1} + u_1s_3^{-1}u_2s_3^{-1}u_1 \\ &\subset s_1^{-1}(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}) + A_3sh(A_3)A_3 \\ &\subset s_1^{-1}(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}) + U_0\end{aligned}$$

by Lemma 2.3. Since s_1^{-1} generates A_2 this proves (1).

We now prove (2). We have

$$\begin{aligned}(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})s_2^{-1} &= s_3^{-1}(s_2s_1^{-1}s_2)s_3^{-1}s_2^{-1} \\ &\in cs_3^{-1}(s_2^{-1}s_1s_2^{-1})s_1^{-1}s_3^{-1}s_2^{-1} + s_3^{-1}u_1u_2u_1s_3^{-1}s_2^{-1} \\ &\subset cs_3^{-1}(s_2^{-1}s_1s_2^{-1})s_1^{-1}s_3^{-1}s_2^{-1} + U_0\end{aligned}$$

by Lemma 2.4. By Lemma 2.3 it follows that

$$\begin{aligned}&(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})s_2^{-1} && \in cs_3^{-1}s_1^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_2^{-1} + U_0 \\ &= cs_1^{-1}s_3^{-1}s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2^{-1}) + U_0 && = cs_1^{-1}s_3^{-1}s_2^{-1}s_1s_3^{-1}s_2^{-1}s_3^{-1} + U_0 \\ &= cs_1^{-1}(s_3^{-1}s_2^{-1}s_3^{-1}s_1s_2^{-1}s_3^{-1} + U_0 && = cs_1^{-1}s_2^{-1}s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1} + U_0 \\ &\subset cs_1^{-1}s_2^{-1}s_3^{-1}c^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} + U_0 && = s_1^{-1}s_2^{-1}s_3^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} + U_0\end{aligned}$$

again by Lemma 2.4. Since

$$s_1^{-1}s_2^{-1}s_3^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} \in s_1^{-1}s_2^{-1}s_3^{-1}s_1(s_2s_1^{-1}s_2)s_3^{-1} + U_0 \subset s_1^{-1}s_2^{-1}s_1s_3^{-1}(s_2s_1^{-1}s_2)s_3^{-1} + U_0$$

by Lemma 2.3, this proves (2)

Since A_3 is generated by s_1^{-1} and s_2^{-1} and U_0 is an $A_3 - A_3$ submodule of A_4, we need to check (3) only for $x = s_1^{-1}$ and $x = s_2^{-1}$, and we just did. □

Proof of Theorem 4.1.

Since $1 \in U$ and U is an A_3-submodule of A_4, in order to prove (1) one need to prove $s_3U \subset U$. Clearly $U \subset A_3u_3A_3u_3A_3$ hence $s_3U \subset u_3A_3u_3A_3u_3A_3 \subset A_3u_3A_3u_3A_3$ by Lemma 4.3, and $A_3(u_3A_3u_3)A_3 \subset A_3UA_3 = U$ by Lemma 4.2 which proves the claim. Then (2) and (3) are consequences of (1) by Lemma 4.4. This concludes the proof of the theorem.

We now let $w^+ = s_3s_2^{-1}s_1s_2^{-1}s_3$, and $w^- = s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} \in A_4$. We recall that $U_0 = A_3u_3A_3 + A_3u_3u_2u_3A_3 \subset A_4$ is a sub-bimodule, and let $U^+ = A_3w^+ + U_0$.

Let $w_0 = s_3s_2s_1^2s_2s_3$. It is classical that, already in the braid group B_4, w_0 commutes with s_1 and s_2. Thus clearly $A_3w_0A_3 = A_3w_0$ and $A_3w_0^{-1}A_3 = A_3w_0^{-1}$. The lemma below thus provides another explanation of Lemma 4.4 above.

Lemma 4.6. (1) $w_0 \in A_3^{\times}w^+ + U_0$, $w_0^{-1} \in A_3^{\times}w^- + U_0$
(2) $U^+ = A_3w_0 + U_0$
(3) $s_3A_3s_3^{-1} \subset U_0$, $s_3^{-1}A_3s_3 \subset U_0$
(4) $s_3A_3s_3 \subset U^+$.

Proof. We have $w_0 = s_3(s_2s_1^2s_2)s_3 \in cs_3s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1s_2s_3 + Rs_3s_2^2s_3$. Clearly $s_3s_2^2s_3 \in U_0$ and $s_3(s_2s_1s_2)s_3 = s_3(s_1s_2s_1)s_3 = s_1s_3s_2s_3s_1 \in U_0$. Moreover, by Lemmas 2.3 and 2.4, $s_3(s_2s_1^{-1}s_2)s_3 \in cs_3s_1^{-1}(s_2^{-1}s_1s_2^{-1})s_3 + s_3u_1u_2u_1s_3 \subset cs_1^{-1}w^+ + U_0$ and thus $w_0 \in A_3^{\times}w^+ + U_0$. As a consequence, $w_0^{-1} = s_3^{-1}s_2^{-1}s_1^{-2}s_2^{-1}s_3^{-1} = \Phi(w_0) \in \Phi(A_3^{\times})\Phi(w^+) + \Phi(U_0) = A_3^{\times}w^- + U_0$, and this proves (1). By definition we have $U^+ = A_3w^+ + U_0 \subset A_3(A_3^{\times}w_0 + U_0) + U_0 \subset A_3w_0 + U_0$, and conversely $A_3w_0 + U_0 \subset A_3(A_3^{\times}w^+ + U_0) + U_0 \subset U^+$; this proves (2). (3) and (4) are given by the proof of Lemma 4.2. □

An immediate consequence is the following variation on Theorem 4.1.

Theorem 4.7. (1) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w_0A_3 + A_3w_0^{-1}A_3$
(2) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w_0 + A_3w_0^{-1}$
(3) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + w_0A_3 + w_0^{-1}A_3$.

From this one easily gets the following generating set of A_4 as A_3-module. Another generating set can be found in [5] §4B.

Proposition 4.8. *As a left A_3-module, A_4 is generated by the 27 elements*

$$\{1, s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}, s_3s_2^{-1}s_1s_2^{-1}s_3, s_3, s_3^{-1}, s_3^{\pm}s_2^{\pm}, s_3^{\pm}s_2^{\pm}s_1^{\pm}, s_3^{\pm}s_2^{-1}s_1s_2^{-1}, s_3s_2^{-1}s_3, s_3s_2^{-1}s_3s_1^{\pm}, s_3s_2^{-1}s_3s_1s_2^{-1}s_1, s_3s_2^{-1}s_3s_1^{\pm}s_2^{\pm}\}.$$

Proof. We denote S the set of 27 elements of the statement and L its span as an A_3-module. We have $A_4 = A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + A_3s_3s_2^{-1}s_1s_2^{-1}s_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3$ by Theorem 4.1, and clearly $A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + A_3s_3s_2^{-1}s_1s_2^{-1}s_3 \subset L$. Moreover, since $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + A_2s_2^{-1}s_1s_2^{-1}$ we have

$$A_3s_3^{\alpha}A_3 = A_3s_3^{\alpha} + \sum_{\substack{\varepsilon\in\{-1,0,1\}\\ \beta\in\{-1,1\}}} A_3s_3^{\alpha}s_2^{\beta}s_1^{\varepsilon} + A_3s_3^{\alpha}s_2^{-1}s_1s_2^{-1} \subset L$$

for any $\alpha \in \{-1, 1\}$. It remains to prove $A_3s_3s_2^{-1}s_3A_3 \subset L$. Since $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + s_2^{-1}s_1s_2^{-1}A_2$, we have $A_3s_3s_2^{-1}s_3A_3 = A_3s_3s_2^{-1}s_3A_2 + A_3s_3s_2^{-1}s_3A_2s_2^{-1}A_2 + A_3s_3s_2^{-1}s_3A_2s_2A_2 + A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2$. Clearly $A_3s_3s_2^{-1}s_3A_2$ is A_3-spanned by the $s_3s_2^{-1}s_3s_1^{\varepsilon}$ for $\varepsilon \in \{0, 1, -1\}$ hence $A_3s_3s_2^{-1}s_3A_2 \subset L$. Now $s_3s_2^{-1}s_3s_2^{-1} \in s_2^{-1}s_3s_2^{-1}s_3 + u_2u_3 + u_3u_2$ by Lemma 3.5, hence $A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2 \subset A_3s_3s_2^{-1}s_3s_1s_2^{-1}A_2 + A_3u_3u_2s_1s_2^{-1}A_2 + A_3u_3s_1s_2^{-1}A_2$, that is

$$A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2 \subset A_3s_3s_2^{-1}s_3s_1s_2^{-1}A_2 + A_3u_3A_3$$

We thus only need to show $A_3s_3s_2^{-1}s_3A_2s_2^{\beta}A_2 \subset L$ for $\beta \in \{-1, 1\}$. This module is A_3-spanned by the $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_1^{\gamma}$ for $\alpha, \gamma \in \{0, 1, -1\}$. The elements belong to S when $\gamma = 0$, so we can assume $\gamma \in \{-1, 1\}$. When $\alpha = 0$, in case $\beta = 1$ we have $s_3(s_2^{-1}s_3s_2)s_1^{\gamma} = s_3s_3s_2s_3^{-1}s_1^{\gamma} \in u_3s_2s_3^{-1}s_1^{\gamma}$. This latter module is spanned by the $s_3^{\varepsilon}s_2s_3^{-1}s_1^{\gamma}$ for $\varepsilon \in \{-1, 0, 1\}$. In case $\varepsilon = 0$ such an element belongs to $A_3u_3A_3 \subset L$; when $\varepsilon = 1$ we can use $(s_3s_2s_3^{-1})s_1^{\gamma} = s_2^{-1}s_3s_2s_1^{\gamma} \in A_3s_3s_2s_1^{\gamma} \in L$; when $\varepsilon = -1$ we have $s_3^{-1}s_2s_3^{-1}s_1^{\gamma} \in A_3s_3s_2^{-1}s_3s_1^{\gamma}$ by Lemmas 2.3 and 2.4, and $s_3s_2^{-1}s_3s_1^{\gamma} \in S$. We can thus assume $\alpha \neq 0$.

We consider first the case $\gamma = -\alpha$. We have $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_1^{-\alpha} = s_3s_2^{-1}s_3s_2^{-\alpha}s_1^{\beta}s_2^{\alpha}$. Then, either $\alpha = 1$ and, by Lemma 3.5,

$$(s_3s_2^{-1}s_3s_2^{-1})s_1^{\beta}s_2 \in s_2^{-1}s_3s_2^{-1}s_3s_1^{\beta}s_2 + u_2u_3s_1^{\beta}s_2 + u_3u_2s_1^{\beta}s_2 \subset L,$$

or $\alpha = -1$ and $s_3(s_2^{-1}s_3s_2)s_1^{\beta}s_2^{-1} = s_3s_3s_2s_3^{-1}s_1^{\beta}s_2^{-1} \in u_3s_2s_3^{-1}s_1^{\beta}s_2^{-1}$. This latter module is spanned by the $s_3^{\varepsilon}s_2s_3^{-1}s_1^{\beta}s_2^{-1}$ which clearly belong to L for $\varepsilon = 0$ and, because of $s_3s_2s_3^{-1} = s_2^{-1}s_3s_2$, for $\varepsilon = 1$; in case $\varepsilon = -1$ it is readily shown to belong to L by Lemmas 2.3 and 2.4 applied to $s_3^{-1}s_2s_3^{-1}$.

We can now assume $\gamma = \alpha$. In case $\beta = \alpha = \gamma$, we have $s_3s_2^{-1}s_3(s_1^{\alpha}s_2^{\alpha}s_1^{\alpha}) = s_3s_2^{-1}s_3s_2^{\alpha}s_1^{\alpha}s_2^{\alpha}$ and, when $\alpha = 1$ we have $s_3(s_2^{-1}s_3s_2)s_1s_2 = s_3s_3s_2s_3^{-1}s_1s_2 \in u_3s_2s_3^{-1}s_1s_2 \subset L$ by similar arguments as for $u_3s_2s_3^{-1}s_1^{\beta}s_2^{-1}$; when $\alpha = -1$, we have, by Lemma 3.5,

$$(s_3s_2^{-1}s_3s_2^{-1})s_1^{-1}s_2^{-1} \in s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2^{-1} + u_3u_2s_1^{-1}s_2^{-1} + u_2u_3s_1^{-1}s_2^{-1} \subset L$$

We thus only need to consider the $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}$. By Lemmas 2.3 and 2.4, we have $s_1^{-1}s_2s_1^{-1} \in s_2(s_1s_2^{-1}s_1) + u_2u_1u_2$, and $s_3s_2^{-1}s_3u_2u_1u_2$ belongs to the A_3-span of our list by our previous arguments. It follows that it only remains to consider $s_3s_2^{-1}s_3s_1s_2^{-1}\, s_1$, which belongs to our list, and $s_3(s_2^{-1}s_3s_2)s_1s_2^{-1}\, s_1 = s_3^2s_2s_3^{-1}s_1s_2^{-1}\, s_1$, which lies in the linear span of the $s_3^{\varepsilon}s_2s_3^{-1}s_1s_2^{-1}\, s_1$ for $\varepsilon \in \{-1, 0, 1\}$. Clearly this element belongs to L in case $\varepsilon = 0$, when $\varepsilon = 1$ it also belongs to L because of $(s_3s_2s_3^{-1})s_1s_2^{-1}\, s_1 = s_2^{-1}s_3s_2s_1s_2^{-1}\, s_1 \in A_3u_3A_3 \subset L$, and when $\varepsilon = -1$ Lemmas 2.3 and 2.4 applied to $s_3^{-1}s_2s_3^{-1}$ show that

$$s_3^{-1}s_2s_3^{-1}s_1s_2^{-1}\, s_1 \in A_3s_3s_2^{-1}s_3s_1s_2^{-1}\, s_1 + A_3u_3A_3 \subset L,$$

and this concludes the proof. □

For subsequent use we prove here the following lemma.

Lemma 4.9. $w_0^2 \in A_3^{\times}w_0^{-1} + U^+$.

Proof. We have $w_0^2 = s_3s_2(s_1^2)s_2s_3^2s_2s_1^2s_2s_3 \in R^{\times}s_3s_2s_1^{-1}s_2s_3^2s_2s_1^2s_2s_3 + Rs_3s_2s_1s_2s_3^2s_2s_1^2s_2s_3 + Rs_3s_2^2s_3^2s_2s_1^2s_2s_3$, and $s_3(s_2s_1s_2)s_3^2s_2s_1^2s_2s_3 = s_3s_1s_2s_1s_3^2s_2s_1^2s_2s_3 = s_1(s_3s_2s_3)s_3s_1s_2s_1^2s_2s_3 = s_1s_2(s_3s_2s_3)s_1s_2s_1^2s_2s_3 = s_1s_2s_2s_3s_2s_1s_2s_1^2s_2s_3 \in U_0^+$ by Lemma 4.6, while $s_3s_2^2(s_3^2)s_2s_1^2s_2s_3 \in Rs_3s_2^2s_3^{-1}s_2s_1^2s_2s_3 + Rs_3s_2^2s_3s_2s_1^2s_2s_3 + Rs_3s_2^2s_2s_1^2s_2s_3$, clearly $s_3s_2^3s_2s_1^2s_2s_3 \in U^+$ by Lemma 4.6,

$$\begin{aligned} s_3s_2^2s_3s_2s_1^2s_2s_3 &= s_3s_2(s_2s_3s_2)s_1^2s_2s_3 = s_3s_2s_3s_2s_3s_1^2s_2s_3 \\ &= (s_3s_2s_3)s_2s_1^2(s_3s_2s_3) = s_2s_3s_2s_2s_1^2s_2s_3s_2 \quad \in U^+ \end{aligned}$$

by Lemma 4.6, and finally

$$(s_3s_2^2s_3^{-1})s_2s_1^2s_2s_3 = s_2^{-1}(s_3^2)s_2^2s_1^2s_2s_3 \in Rs_2^{-1}s_3^{-1}s_2^2s_1^2s_2s_3 + Rs_2^{-1}s_3s_2^2s_1^2s_2s_3 + Rs_2s_1^2s_2s_3 \subset U^+$$

by Lemma 4.6. Thus, $w_0^2 \in R^\times s_3s_2s_1^{-1}s_2s_3^2s_2s_1^2s_2s_3 + U^+$. Now, $s_3s_2s_1^{-1}s_2s_3^2s_2(s_1^2)s_2s_3 \in R^\times s_3s_2s_1^{-1}s_2s_3^2s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1^{-1}s_2s_3^2s_2s_1s_2s_3 + Rs_3s_2s_1^{-1}s_2s_3^2s_2^2s_3$. We have $s_3s_2s_1^{-1}s_2s_3^2(s_2s_1s_2)s_3 = s_3s_2s_1^{-1}s_2s_3^2s_1s_2s_1s_3 = s_3s_2(s_1^{-1}s_2s_1)s_3^2s_2s_3s_1 = s_3s_2^2s_1(s_2^{-1}s_3^2s_2)s_3s_1 = s_3s_2^2s_1s_3s_2^2s_3^{-1}s_3s_1 = s_3s_2^2s_1s_3s_2^2s_1 \in U^+$ by Lemma 4.6. Now $s_3s_2s_1^{-1}s_2(s_3^2)s_2^2s_3 \in Rs_3s_2s_1^{-1}s_2s_3^{-1}s_2^2s_3 + Rs_3s_2s_1^{-1}s_2s_3s_2^2s_3 + Rs_3s_2s_1^{-1}s_2^3s_3$; we have $s_3s_2s_1^{-1}s_2^3s_3 \in U^+$ by Lemma 4.6, $s_3s_2s_1^{-1}s_2s_3s_2^2s_3 = s_3s_2s_1^{-1}(s_2s_3s_2)s_2s_3 = s_3s_2s_1^{-1}s_3s_2s_3s_2s_3 = (s_3s_2s_3)s_1^{-1}s_2(s_3s_2s_3) = s_2s_3s_2s_1^{-1}s_2s_2s_3s_2 \in U^+$ by Lemma 4.6, and $s_3s_2s_1^{-1}s_2(s_3^{-1}s_2^2s_3) = s_3s_2s_1^{-1}s_2s_2s_3^2s_2^{-1} \in s_3s_2s_1^{-1}s_2^2s_2(R + Rs_3 + Rs_3^{-1})s_2^{-1} \subset U^+$ by Lemma 4.6.

Thus $w_0^2 \in R^\times s_3s_2s_1^{-1}s_2s_3^2s_2s_1^{-1}s_2s_3 + U^+$. Now,

$$s_3s_2s_1^{-1}s_2(s_3^2)s_2s_1^{-1}s_2s_3 \in R^\times s_3s_2s_1^{-1}s_2s_3^{-1}s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1^{-1}s_2s_3s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1^{-1}s_2^2s_1^{-1}s_2s_3$$

We have $s_3s_2s_1^{-1}(s_2s_3s_2)s_1^{-1}s_2s_3 = s_2s_3s_2s_1^{-1}s_2s_1^{-1}s_2s_3s_2 \in U^+$ by Lemma 4.6, $s_3s_2s_1^{-1}s_2^2s_1^{-1}s_2s_3 \in U^+$ by Lemma 4.6, hence $w_0^2 \in R^\times s_3s_2s_1^{-1}s_2s_3^{-1}s_2s_1^{-1}s_2s_3 + U^+$. Using $s_2s_1^{-1}s_2 \in A_2^\times s_2^{-1}s_1s_2^{-1} + u_1u_2u_1$ (see Lemmas 2.4 and 2.3),we get $s_3(s_2s_1^{-1}s_2)s_3^{-1}s_2s_1^{-1}s_2s_3 \in A_2^\times s_3s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2s_1^{-1}s_2s_3 + s_3u_1u_2u_1s_3^{-1}s_2s_1^{-1}s_2s_3$. Since

$$s_3u_1u_2u_1s_3^{-1}s_2s_1^{-1}s_2s_3 = u_1(s_3u_2s_3^{-1})u_1s_2s_1^{-1}s_2s_3 \subset u_1s_2^{-1}u_3s_2u_1s_2s_1^{-1}s_2s_3 \subset U^+$$

by Lemma 4.6, we have $w_0^2 \in A_2^\times s_3s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2s_1^{-1}s_2s_3 + U^+$. Now $s_3s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2)s_1^{-1}s_2s_3 = s_3s_2^{-1}s_3s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1}$ and, using $s_3s_2^{-1}s_3 \in u_2^\times s_3^{-1}s_2s_3^{-1} + u_2u_3u_2$, we get

$$s_3s_2^{-1}s_3s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} \in u_2^\times s_3^{-1}s_2s_3^{-1}s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} + u_2u_3u_2s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1}$$

We have $u_2u_3u_2s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} \in U^+$ by Lemma 4.6, and

$$s_3^{-1}s_2s_3^{-1}s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} = s_3^{-1}s_2s_1^2(s_3^{-1}s_2^{-1}s_3)s_1^{-1}s_2^{-1} = s_3^{-1}s_2s_1^2s_2s_3^{-1}s_2^{-1}s_1^{-1}s_2^{-1}$$

Thus $w_0^2 \in A_3^\times s_3^{-1}s_2s_1^2s_2s_3^{-1}(s_2^{-1}s_1^{-1}s_2^{-1}) + U^+$. Since $s_3^{-1}s_2(s_1^2)s_2s_3^{-1} \in R^\times s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + Rs_3^{-1}s_2s_1s_2s_3^{-1} + Rs_3^{-1}s_2^2s_3^{-1}$ and clearly $s_3^{-1}s_2^2s_3^{-1} \in U_0$, $s_3^{-1}(s_2s_1s_2)s_3^{-1} = s_3^{-1}s_1s_2s_1s_3^{-1} = s_1s_3^{-1}s_2s_3^{-1}s_1 \in U_0$, we have $w_0^2 \in A_3^\times w^-(s_2^{-1}s_1^{-1}s_2^{-1}) + U^+$, hence $w_0^2 \in A_3^\times w_0^{-1}(s_2^{-1}s_1^{-1}s_2^{-1}) + U^+$ by Lemma 4.6(1). Since w_0 commutes with s_1 and s_2 this yields $w_0^2 \in A_3^\times w_0^{-1} + U^+$. □

5. The algebra A_4 as a $\langle s_1, s_3\rangle$ (bi)module

Let $B = \langle s_1, s_3\rangle$ denote the subalgebra (with 1) of A_4 generated by s_1 and s_3. In order to describe A_5 as an A_4-module we will need the description of A_4 as a B-module, that we do in this section. Note that this will provide another proof of the conjecture of [6] for A_4.

First note that there are *three* automorphisms or skew-automorphisms of the pair (A_4, B): in addition to the automorphism Φ and the skew-automorphism Ψ, there is the automorphism $\mathrm{Ad}\,\Delta : x \mapsto \Delta x\Delta^{-1}$, where Δ is the image in A_4 of Garside's Δ in the braid group on 4 strands, that is $\Delta = s_1s_2s_3s_1s_2s_1 = s_1(s_2s_3s_1s_2)s_1$; this automorphism exchanges s_1 and s_3 and fixes s_2.

We denote $A_4^{[0]} = B$, $A_4^{[n+1]} = A_4^{[n]}u_2B = A_4^{[n]} + A_4^{[n]}s_2B + A_4^{[n]}s_2^{-1}B$, and in particular $A_4^{[1]} = B + Bs_2B + Bs_2^{-1}B$. We first prove several lemmas.

Lemma 5.1. (1) *For $i, j \in \{1, 3\}$ we have $u_2u_iu_2u_ju_2 \subset A_4^{[2]}$.*
(2) *For $i, j, k \in \{1, 3\}$ we have $u_2u_iu_2u_ju_ku_2 \subset A_4^{[2]}$ and $u_2u_iu_ju_2u_ku_2 \subset A_4^{[2]}$.*

Proof. We prove (1). If $i = j$, up to applying $\mathrm{Ad}\,\Delta$ we can assume $i = j = 1$ and the statement is a consequence of the study of A_3, as $u_2u_1u_2u_1u_2 \subset A_3 \subset u_1u_2u_1u_2 + u_1u_2u_1$. Thus we can assume $i \neq j$, and by using $\mathrm{Ad}\,\Delta$ and Ψ we only need to consider $X = s_2^\alpha s_1^\beta s_2^\gamma s_3^\delta s_2^\varepsilon$ with $\alpha, \dots, \varepsilon \in \{-1, 1\}$. If $\alpha = -\gamma$ or $\gamma = -\varepsilon$, then we get $X \in A_4^{[2]}$ by using $s_2^\alpha s_1^\beta s_2^{-\alpha} = s_1^{-\alpha}s_2^\beta s_1^\alpha$ and $s_2^\gamma s_3^\delta s_2^{-\gamma} = s_3^{-\gamma}s_2^\delta s_3^\gamma$. Up to applying Φ we can thus assume $\alpha = \gamma = \varepsilon = 1$, that is $X = s_2s_1^\beta s_2s_3^\delta s_2$. If $\beta = 1$ or $\delta = 1$ we get $X \in A_4^{[2]}$ by $s_2s_1s_2 = s_1s_2s_1$ and $s_2s_3s_2 = s_3s_2s_3$. One can thus assume $X = s_2s_1^{-1}s_2s_3^{-1}s_2$. By Lemmas 2.4 and 2.3 we have $s_2s_1^{-1}s_2 \in u_1^\times s_2^{-1}s_1s_2^{-1} + u_1u_2u_1$ hence $X \in u_1^\times s_2^{-1}s_1s_2^{-1}s_3s_2 + A_4^{[2]}$ and $s_2^{-1}s_1(s_2^{-1}s_3s_2) = s_2^{-1}s_1s_3s_2s_3^{-1} \in A_4^{[2]}$, and this concludes the proof of (1).

We prove (2). Up to applying Ψ we can confine ourselves to prove $u_2u_iu_2u_ju_ku_2 \subset A_4^{[2]}$. By (1) and $u_j^2 = u_j$, $u_k^2 = u_k$ we can assume $j \neq k$, that is $\{j, k\} = \{1, 3\}$. Up to applying $\mathrm{Ad}\,\Delta$ we can assume $i = 1$, hence we want to prove $u_2u_1u_2u_1u_3u_2 \subset A_4^{[2]}$. We have $u_2u_1u_2u_1 \subset A_3 = u_1u_2u_1u_2 + u_1u_2u_1$ hence $u_2u_1u_2u_1u_3u_2 \subset u_1u_2u_1u_2u_3u_2 + u_1u_2u_1u_3u_2 \subset A_4^{[2]}$ by (1). □

Lemma 5.2.

$$A_4^{[3]} \subset A_4^{[2]} + \sum_{\alpha,\beta\in\{-1,1\}} Bs_2^\alpha(s_1s_3^{-1})^\beta s_2^\alpha(s_1s_3^{-1})^\beta s_2^\alpha B + \sum_{\alpha,\beta\in\{-1,1\}} Bs_2^\alpha(s_1s_3)^\beta s_2^{-\beta}(s_1s_3)^\beta s_2^\varepsilon B$$

Proof. We only need to prove that all the terms of the form $s_2^{\alpha}s_1^{\beta_1}s_3^{\beta_3}s_2^{\gamma}s_1^{\delta_1}s_3^{\delta_3}s_2^{\varepsilon}$ belong to the RHS, as all the over natural linear generators of $A_4^{[3]}$ belong to $A_4^{[2]}$ by Lemma 5.1. We remark that the RHS is stable under Φ, Ψ and $\mathrm{Ad}\,\Delta$. We first assume $\beta_1 = -\delta_1$. Then

$$s_2^{\alpha}s_1^{\beta_1}s_3^{\beta_3}s_2^{\gamma}s_1^{\delta_1}s_3^{\delta_3}s_2^{\varepsilon} = s_2^{\alpha}s_3^{\beta_3}(s_1^{\beta_1}s_2^{\gamma}s_1^{-\beta_1})s_3^{\delta_3}s_2^{\varepsilon} = s_2^{\alpha}s_3^{\beta_3}s_2^{-\beta_1}s_1^{\gamma}s_2^{\beta_1}s_3^{\delta_3}s_2^{\varepsilon}$$

If $\alpha = \beta_1$ or $\varepsilon = -\beta_1$, such a term belongs to $A_4^{[2]}$ by $s_2^{\alpha}s_3^{\beta_3}s_2^{-\alpha} = s_3^{-\alpha}s_2^{\beta_3}s_3^{\alpha}$ or $s_2^{-\varepsilon}s_3^{\delta}s_2^{\varepsilon} = s_3^{\varepsilon}s_2^{\delta}s_3^{-\varepsilon}$ and Lemma 5.1. We thus only need to consider $X = s_2^{-\beta_1}s_3^{\beta_3}s_2^{-\beta_1}s_1^{\gamma}s_2^{\beta_1}s_3^{\delta_3}s_2^{\beta_1}$. Since $s_2^{-\beta_1}s_3^{-\beta_1}s_2^{-\beta_1} = s_3^{-\beta_1}s_2^{-\beta_1}s_3^{-\beta_1}$ and $s_2^{\beta_1}s_3^{\beta_1}s_2^{\beta_1} = s_3^{\beta_1}s_2^{\beta_1}s_3^{\beta_1}$, by Lemma 5.1 we can assume $\beta_3 = \beta_1$ and $\delta = -\beta_1$, that is $X = s_2^{-\beta_1}s_3^{\beta_1}s_2^{-\beta_1}s_1^{\gamma}s_2^{\beta_1}s_3^{-\beta_1}s_2^{\beta_1}$. By applying Φ and Ψ we can assume $X = s_2s_3^{-1}s_2s_1s_2^{-1}s_3s_2^{-1}$. By Lemma 2.4 we have $s_2^{-1}s_3s_2^{-1} \in s_2s_3^{-1}s_2u_3^{\times} + u_3u_2u_3$ hence $s_2s_3^{-1}s_2s_1(s_2^{-1}s_3s_2^{-1}) \in s_2s_3^{-1}s_2s_1s_2s_3^{-1}s_2u_3^{\times} + s_2s_3^{-1}s_2s_1u_3u_2u_3$. We have $s_2s_3^{-1}s_2s_1u_3u_2u_3 \subset A_4^{[2]}$ by Lemma 5.1 and $s_2s_3^{-1}s_2s_1s_2s_3^{-1}s_2$ belongs to the RHS, which concludes this case.

The case $\beta_3 = -\delta_3$ is a consequence of the previous case by applying $\mathrm{Ad}\,\Delta$. We thus only need to consider $X = s_2^{\alpha}s_3^{\beta_3}s_1^{\beta_1}s_2^{\gamma}s_1^{\beta_1}s_3^{\beta_3}s_2^{\varepsilon}$. First assume $\gamma = \beta_1$. Then $X = s_2^{\alpha}s_3^{\beta_3}(s_1^{\gamma}s_2^{\gamma}s_1^{\gamma})s_3^{\beta_3}s_2^{\varepsilon} = s_2^{\alpha}s_3^{\beta_3}s_2^{\gamma}s_1^{\gamma}s_2^{\gamma}s_3^{\beta_3}s_2^{\varepsilon}$ belongs as before to $A_4^{[2]}$ by Lemma 5.1 and elementary transformations, unless $\varepsilon = \gamma$, $\alpha = \gamma$, and then $\beta_3 = -\gamma$. In that case $X = s_2^{\alpha}s_3^{-\alpha}(s_2^{\alpha}s_1^{\alpha}s_2^{\alpha})s_3^{-\alpha}s_2^{\alpha} = s_2^{\alpha}s_3^{-\alpha}s_1^{\alpha}s_2^{\alpha}s_1^{\alpha}s_3^{-\alpha}s_2^{\alpha}$ belongs to the RHS. We can thus assume $\gamma \neq \beta_1$ and, applying $\mathrm{Ad}\,\Delta$, $\gamma \neq \beta_3$, hence we can assume $\beta_1 = \beta_3 = -\gamma$. Then $X = s_2^{\alpha}s_3^{\gamma}s_1^{\gamma}s_2^{-\gamma}s_1^{\gamma}s_3^{\gamma}s_2^{\varepsilon}$, which belongs to the RHS, and this concludes the proof. $\square$

Lemma 5.3. *Let $\alpha, \beta, \varepsilon \in \{-1, 1\}$. Then $s_2^{\alpha}(s_1s_3)^{\beta}s_2^{-\beta}(s_1s_3)^{\beta}s_2^{\varepsilon}$ belongs to*

$$A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs_2^{\delta}(s_1s_3)^{\delta}s_2^{-\delta}(s_1s_3)^{\delta}s_2^{\delta}B + \sum_{\delta\in\{-1,1\}} Bs_2^{-\delta}(s_1s_3)^{\delta}s_2^{-\delta}(s_1s_3)^{\delta}s_2^{-\delta}B$$

Proof. First assume $\alpha = \beta$. Then $X = s_2^{\beta}s_1^{\beta}s_3^{\beta}s_2^{-\beta}s_3^{\beta}(s_1^{\beta}s_2^{\varepsilon}s_1^{-\beta})s_1^{\beta} = s_2^{\beta}s_1^{\beta}(s_3^{\beta}s_2^{-\beta}s_3^{\beta}s_2^{-\beta})s_1^{\varepsilon}s_2^{\beta}s_1^{\beta} \in s_2^{\beta}s_1^{\beta}s_2^{-\beta}s_3^{\beta}s_2^{-\beta}s_3^{\varepsilon}s_1^{\varepsilon}s_2^{\beta}s_1^{\beta} + s_2^{\beta}s_1^{\beta}u_2u_3s_1^{\varepsilon}s_2^{\beta}s_1^{\beta} + s_2^{\beta}s_1^{\beta}u_3u_2s_1^{\varepsilon}s_2^{\beta}s_1^{\beta}$ by Lemma 3.5. Now $s_2^{\beta}s_1^{\beta}u_2u_3s_1^{\varepsilon}s_2^{\beta}s_1^{\beta} \subset A_4^{[2]}$ and $s_2^{\beta}s_1^{\beta}u_3u_2s_1^{\varepsilon}s_2^{\beta}s_1^{\beta} \subset A_4^{[2]}$ by Lemma 5.1. We thus only need to consider

$$X = (s_2^{\beta}s_1^{\beta}s_2^{-\beta})s_3^{\beta}s_2^{-\beta}s_3^{\beta}s_1^{\varepsilon}s_2^{\beta} = s_1^{-\beta}s_2^{\beta}s_1^{\beta}s_3^{\beta}s_2^{-\beta}s_3^{\beta}s_1^{\varepsilon}s_2^{\beta}$$

hence, if $\varepsilon = -\beta$, we get

$$X = s_1^{-\beta}.s_2^{\beta}s_3^{\beta}(s_1^{\beta}s_2^{-\beta}s_1^{-\beta})s_3^{\beta}s_2^{\beta} = s_1^{-\beta}.s_2^{\beta}s_3^{\beta}s_2^{-\beta}s_1^{-\beta}(s_2^{\beta}s_3^{\beta}s_2^{\beta}) = s_1^{-\beta}.s_2^{\beta}s_3^{\beta}s_2^{-\beta}s_1^{-\beta}s_3^{\beta}s_2^{\beta}s_3^{\beta} \in A_4^{[2]}$$

by Lemma 5.1. We can thus assume $\varepsilon = \beta$, in which case X clearly belongs to the space we want.

This concludes the case $\alpha = \beta$, hence also the case $\varepsilon = \beta$ by application of Φ and Ψ. Thus we can assume $\alpha = -\beta = \varepsilon$, and the conclusion is clear in this case. $\square$

Lemma 5.4. *For $\alpha, \beta \in \{-1, 1\}$, we have*

$$s_2^{\alpha}(s_1s_3^{-1})^{\beta}s_2^{\alpha}(s_1s_3^{-1})^{\beta}s_2^{\alpha} \in A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs_2^{\delta}(s_1s_3)^{\delta}s_2^{-\delta}(s_1s_3)^{\delta}s_2^{\delta}B$$

Proof. The RHS is stable under $\mathrm{Ad}\,\Delta$ and Φ, hence we can assume $\alpha = \beta = 1$, and thus we only need to consider $X = s_2s_1s_3^{-1}s_2s_1s_3^{-1}s_2 = s_2s_1(s_3^{-1}s_2s_3^{-1})s_1s_2 \in s_2s_1u_2s_3s_2^{-1}s_3s_1s_2 + s_2s_1u_2u_3u_2s_1s_2$ by Lemmas 2.4 and 2.3. We have

$$s_2s_1u_2u_3u_2s_1s_2 \subset \sum_{a\in\{0,1,-1\}} s_2s_1s_2^{a}u_3u_2s_1s_2$$

and,

- if $a = 0$ we have $s_2s_1u_3u_2s_1s_2 \subset A_4^{[2]}$ by Lemma 5.1;
- if $a = 1$ we have $(s_2s_1s_2)u_3u_2s_1s_2 = s_1s_2s_1u_3u_2s_1s_2 \subset A_4^{[2]}$ by Lemma 5.1;
- if $a = -1$ we have $(s_2s_1s_2^{-1})u_3u_2s_1s_2 = s_1^{-1}s_2s_1u_3u_2s_1s_2 \subset A_4^{[2]}$ by Lemma 5.1

hence $X \in s_2s_1u_2s_3s_2^{-1}s_3s_1s_2 + A_4^{[2]}$. The module $s_2s_1u_2s_3s_2^{-1}s_3s_1s_2$ is R-spanned by the $Y(a) = s_2s_1s_2^{a}s_3s_2^{-1}s_3s_1s_2$ for $a \in \{-1, 0, 1\}$. We have $Y(0) = s_2s_1s_3s_2^{-1}s_3s_1s_2 \in RHS$, $Y(1) = (s_2s_1s_2)s_3s_2^{-1}s_3s_1s_2 = s_1s_2s_1s_3s_2^{-1}s_3s_1s_2 \in RHS$ and

$$Y(-1) = (s_2s_1s_2^{-1})s_3s_2^{-1}s_3s_1s_2 = s_1^{-1}s_2s_1s_3s_2^{-1}s_3s_1s_2 \in RHS,$$

and this concludes the proof. $\square$

In the braid group on 4 strands , we have

$$\Delta = (s_1s_2s_3)(s_1s_2)s_1 = (s_1s_3)(s_2s_1s_3s_2) = (s_2s_1s_3s_2)(s_1s_3)$$

hence the same equalities hold in A_4. In the remaining part of this section, we let $s = s_2$, $p = s_1s_3 = s_3s_1$, hence $\Delta = spsp = psps$. Note that $\Delta p = p\Delta$, $\Delta s = s\Delta$. It follows that $\Delta^2 = p.sps.\Delta = p.\Delta.sps = p(psps)sps = p^2.sps^2ps$, $\Delta^3 = p^2.sps^2ps.\Delta = p^2.\Delta.sps^2ps = p^3.sps^2ps^2p$, and $\Delta^4 = p^4.sps^2ps^2ps^2ps$.

We thus have $\Delta^2 = p^2.sps^2ps$ hence $p^{-2}\Delta^2 \in R^\times sps^{-1}ps + Rspsps + Rsp^2s$ and we known $sp^2s \in A_4^{[2]}$, $(spsp)s = psps^2 \in A_4^{[2]}$ by Lemma 5.1. It follows that

$$(*)\ \ p^{-2}\Delta^2 \in R^\times sps^{-1}ps + Rpsps^2 + Rsp^2s$$
$$p^{-2}\Delta^2 \in R^\times sps^{-1}ps + A_4^{[2]}$$

Applying Φ, we have $\Phi(\Delta) = \Phi(s_1s_2s_3s_1s_2s_1) = s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}s_2^{-1}s_1^{-1} = (s_1s_2s_1s_3s_2s_1)^{-1} = \Delta^{-1}$, hence, since $\Phi(p) = p^{-1}$,

$$(*)\ \ p^{-2}\Delta^2 \in R^\times s^{-1}p^{-1}sp^{-1}s^{-1} + A_4^{[2]}$$

Lemma 5.5. (1) $s_2^{-1}ps_2^{-1}ps_2^{-1}.s_1^{-1} \in u_1^\times s_2p^{-1}s_2p^{-1}s_2 + A_4^{[2]}$
(2) $s_2^{-1}ps_2^{-1}ps_2^{-1}B \subset Bs_2p^{-1}s_2p^{-1}s_2 + Bs_2^{-1}ps_2^{-1}ps_2^{-1} + A_4^{[2]}$
(3) $s_2p^{-1}s_2p^{-1}s_2B \subset Bs_2p^{-1}s_2p^{-1}s_2 + Bs_2^{-1}ps_2^{-1}ps_2^{-1} + A_4^{[2]}$.

Proof. $X = s_2^{-1}ps_2^{-1}ps_2^{-1}.s_1^{-1} = s_2^{-1}ps_2^{-1}s_3(s_1s_2^{-1}.s_1^{-1}) = s_2^{-1}ps_2^{-1}s_3s_2^{-1}s_1^{-1}s_2 = s_2^{-1}s_1(s_3s_2^{-1}s_3s_2^{-1})s_1^{-1}s_2 \in s_2^{-1}s_1s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2 + s_2^{-1}s_1u_2u_3s_1^{-1}s_2 + s_2^{-1}s_1u_3u_2s_1^{-1}s_2$ by Lemma 3.5. We have $s_2^{-1}s_1u_2u_3s_1^{-1}s_2 \subset A_4^{[2]}$ and $s_2^{-1}s_1u_3u_2s_1^{-1}s_2 \subset A_4^{[2]}$ by Lemma 5.1, hence

$$\begin{aligned} X &\in (s_2^{-1}s_1s_2^{-1})s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \\ &\subset u_1^\times s_2s_1^{-1}(s_2s_3s_2^{-1}s_3s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \\ &\subset u_1^\times s_2s_1^{-1}s_3^{-1}s_2s_3^2s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \\ &\subset u_1^\times s_2s_1^{-1}s_3^{-1}s_2p^{-1}s_2 + u_1s_2s_1^{-1}s_3^{-1}s_2s_3s_1^{-1}s_2 + u_1s_2s_1^{-1}s_3^{-1}s_2s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \end{aligned}$$

We know $s_2s_1^{-1}s_3^{-1}s_2s_1^{-1}s_2 \in A_4^{[2]}$ by Lemma 5.1, $s_2s_1^{-1}(s_3^{-1}s_2s_3)s_1^{-1}s_2 = s_2s_1^{-1}s_2s_3(s_2^{-1}s_1^{-1}s_2) = s_2s_1^{-1}s_2s_3s_1s_2^{-1}s_1^{-1} \in A_4^{[2]}$ by Lemma 5.1, and $u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 = u_2s_3u_1s_2^{-1}s_1^{-1}s_3s_2$ is the sum of $u_2s_3s_2^{-1}s_1^{-1}s_3s_2 \subset A_4^{[2]}$ (by Lemma 5.1) and of the $u_2s_3s_1^\alpha s_2^{-1}s_1^{-1}s_3s_2$ for $\alpha \in \{-1, 1\}$. Since $u_2s_3(s_1^\alpha s_2^{-1}s_1^{-1})s_3s_2 = u_2s_3s_2^{-1}s_1^{-1}(s_2^\alpha s_3s_2) = u_2s_3s_2^{-1}s_1^{-1}s_3s_2s_3^\alpha \subset A_4^{[2]}$ by Lemma 5.1, and this proves (1). To get (2) from (1), we use $s_2^{-1}ps_2^{-1}ps_2^{-1}.s_3^{-1} \in u_3^\times s_2p^{-1}s_2p^{-1}s_2^{-1} + A_4^{[2]}$, that we get from (1) by applying Ad Δ, and the fact that B is generated as a unital R-algebra by s_1^{-1} and s_3^{-1}. This proves (2), and then (3) follows from (2) by a direct application of Φ. □

From all this we deduce the following.

Lemma 5.6. (1) $A_4^{[3]} = A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs^\delta p^\delta s^{-\delta}p^\delta s^\delta + \sum_{\delta\in\{-1,1\}} Bs^{-\delta}p^\delta s^{-\delta}p^\delta s^{-\delta}$
(2) $A_4 = A_4^{[3]}$.

Proof. As a consequence of Lemmas 5.2 and 5.3, we get

$$A_4^{[3]} = A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs^\delta p^\delta s^{-\delta}p^\delta s^\delta B + \sum_{\delta\in\{-1,1\}} Bs^{-\delta}p^\delta s^{-\delta}p^\delta s^{-\delta}B$$

We know $s^{-\delta}p^\delta s^{-\delta}p^\delta s^{-\delta}B \subset A_4^{[2]} + \sum_{\varepsilon\in\{-1,1\}} Bs^{-\varepsilon}p^\varepsilon s^{-\varepsilon}p^\varepsilon s^{-\varepsilon}$ by Lemma 5.5 hence

$$A_4^{[3]} = A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs^{-\delta}p^\delta s^{-\delta}p^\delta s^{-\delta} + \sum_{\delta\in\{-1,1\}} Bs^\delta p^\delta s^{-\delta}p^\delta s^\delta B$$

and finally $s^\delta p^\delta s^{-\delta}p^\delta s^\delta \in R^\times p^{-\delta}\Delta^{2\delta} + A_4^{[2]}$ by (*), hence $s^\delta p^\delta s^{-\delta}p^\delta s^\delta B \subset p^{-\delta}\Delta^{2\delta}B + A_4^{[2]} = p^{-\delta}B\Delta^{2\delta} + A_4^{[2]} = B\Delta^{2\delta} + A_4^{[2]} \subset Bs^\delta p^\delta s^{-\delta}p^\delta s^\delta + A_4^{[2]}$ and this concludes the proof of (1). Now $A_4^{[3]}$ is an R-submodule of A_4 which contains 1, which is stable under right-multiplication by s_1 and s_3 by definition. Moreover, in view of (1), we have

$$A_4^{[3]}s_2 \subset A_4^{[2]}s + \sum_{\delta\in\{-1,1\}} Bs^\delta p^\delta s^{-\delta}p^\delta s^\delta s + \sum_{\delta\in\{-1,1\}} Bs^{-\delta}p^\delta s^{-\delta}p^\delta s^{-\delta}s \subset A_4^{[3]}$$

hence $A_4^{[3]}$ is also stable under right multiplication by s_2, hence it is a right-ideal of A_4 containing 1, hence (2). □

We let $x_\pm = s^\pm p^\pm s^\mp p^\pm s^\pm$ and $y_\pm = s^\pm p^\mp s^\pm p^\mp s^\pm$.

Lemma 5.7. (1) $sBsps \subset A_4^{[2]}$
(2) $sBs^{-1}ps \subset Rsps^{-1}psA_4^{[2]}$.

Proof. The R-module $sBsps$ is spanned by $s^2ps \in A_4^{[2]}$, the $ss_isps \in A_4^{[2]}$ for $i \in \{1,3\}$ by Lemma 5.1, $s(psps) = s(spsp) = s^2psp \in A_4^{[2]}$, $s_2s_1(s_3^{-1}s_2s_3)s_1s_2 = s_2s_1s_2s_3(s_2^{-1}s_1s_2) = s_2s_1s_2s_3s_1s_2s_1^{-1} \in A_4^{[2]}$ by Lemma 5.1, the image of this latest one by $\mathrm{Ad}\,\Delta$, and by

$$s_2s_1^{-1}s_3^{-1}s_2s_3s_1s_2 = s_2s_1^{-1}(s_3^{-1}s_2s_3)s_1s_2 = s_2s_1^{-1}s_2s_3(s_2^{-1}s_1s_2) = s_2s_1^{-1}s_2s_3s_1s_2s_1^{-1} \in A_4^{[2]}$$

by Lemma 5.1, and this proves (1).

Now $sBs^{-1}ps$ is R-spanned by $sps^{-1}ps$ and

- the $ss^{-1}ps = ps \in A_4^{[2]}$
- the $ss_is^{-1}ps \in A_4^{[2]}$ for $i \in \{1,3\}$ by Lemma 5.1
- $s_2s_1(s_3^{-1}s_2^{-1}s_3)s_1s_2 = s_2s_1s_2s_3^{-1}(s_2^{-1}s_1s_2) = s_2s_1s_2s_3^{-1}s_1s_2s_1^{-1} \in A_4^{[2]}$ for $i \in \{1,3\}$ by Lemma 5.1
- $\Delta.s_2s_1s_3^{-1}s_2^{-1}s_3s_1s_2\Delta^{-1} \in A_4^{[2]}$
- $s_2s_1^{-1}(s_3^{-1}s_2^{-1}s_3)s_1s_2 = s_2s_1^{-1}s_2s_3^{-1}(s_2^{-1}s_1s_2) = s_2s_1^{-1}s_2s_3^{-1}s_1s_2s_1^{-1} \in A_4^{[2]}$ for $i \in \{1,3\}$ by Lemma 5.1

and this proves (2). $\square$

We want to express Δ^3 in terms of the $x_\pm$ and $y_\pm$. We recall that $\Delta^2 \in R^\times p^2sps^{-1}ps + Rp^3sps^2 + Rp^2sp^2s$ hence $\Delta^3 \in R^\times p^2sps^{-1}ps\Delta + Rp^3sps^2\Delta + Rp^2sp^2s\Delta$. We have

- $sps^{-1}ps\Delta \in Rsp\Delta + Rsps\Delta + Rsps^{-1}\Delta$ and
  (1) $sps^{-1}\Delta = sps^{-1}(spsp) = sp^2sp \in A_4^{[2]}$,
  (2) $sps\Delta = sps(spsp) = sps^2psp \in R^\times sps^{-1}psp + Rspspsp + Rsp^2sp$, and we have $(spsp)sp = psps^2p \in A_4^{[2]}$, $sp^2sp \in A_4^{[2]}$, hence $sps\Delta \in R^\times sps^{-1}psp + A_4^{[2]}$.
  (3) $sps^{-1}\Delta = sps^{-1}spsp = sp^2sp \in A_4^{[2]}$

  hence $sps^2\Delta \in R^\times sps^{-1}psp + A_4^{[2]}$.
- $sp^2s\Delta = sp^2s^2psp \in R^\times sp^2s^{-1}psp + Rsp^2spsp + Rsp^2psp$, and $sp^2(spsp) = sp^2psps = sp^3sps, sp^2psp \in A_4^{[2]}$.

It follows that $\Delta^3 \in R^\times p^2sp^2sp^2s + Rp^3sps^{-1}psp + Rp^2sp^2s^{-1}psp + Rp^2sp^3sps + A_4^{[2]}$. From (*) we have $p^2sps^{-1}ps \in \Delta^2 + A_4^{[2]}$ hence $p^3sps^{-1}psp \in p\Delta^2p + A_4^{[2]} = p^2\Delta^2 + A_4^{[2]}$ hence $p^3sps^{-1}psp \in R^\times p^4.sps^{-1}ps + A_4^{[2]}$. By Lemma 5.7, we have $sp^2s^{-1}ps \in Rx_+ + A_4^{[2]}$ hence $p^2sp^2s^{-1}psp \in Rp^2x_+p + A_4^{[2]} \subset Bx_+ + A_4^{[2]}$. Since $sp^3sps \in A_4^{[2]}$ this leads to

$$\Delta^3 \in R^\times p^2sp^2sp^2s + Bx_+ + A_4^{[2]}$$

Since $s_i^2 = as_i + b + cs_i^{-1}$ we have $p^2 = s_1^2s_3^2 = (as_1 + b + cs_1^{-1})(as_3 + b + cs_3^{-1}) \in a^2p + c^2p^{-1} + W$ with W the R-span of $s_1s_3^{-1}, s_3s_1^{-1}, s_1, s_3, s_1^{-1}, s_3^{-1}, 1$. After easy applications of Lemma 5.1 it follows that $sp^2sp^2s \in c^4sp^{-1}sp^{-1}s + RspsBs + RsBsps + A_4^{[2]}$. Since $spsBs + sBsps \subset A_4^{[2]}$ by Lemma 5.7 we get $\Delta^3 \in c^4p^2y_+ + Bx_+ + A_4^{[2]}$ and $\Delta^{-3} = \Phi(\Delta^3) \in c^{-4}p^{-2}y_- + Bx_+ + A_4^{[2]}$. Now we have $\Delta^3s_1 = s_3\Delta^3 \in c^4s_3p^2y_+ + Bx_+ + A_4^{[2]}$ and $\Delta^3s_1 \in c^4p^2y_+s_1 + Bx_+B + A_4^{[2]}$, $\Delta^3s_1 \in c^4p^2u_1^\times y_- + Bx_+B + A_4^{[2]}$ by Lemma 5.5 (1), $\Delta^3s_1 \in c^4p^2u_1^\times y_- + Bx_+ + A_4^{[2]}$ by using $p^{-2}\Delta^2 \in R^\times x_+ + A_4^{[2]}$. It follows that $c^4s_3p^2y_+ \in c^4p^2u_1^\times y_- + Bx_+ + A_4^{[2]}$ hence $y_+ \in By_- + Bx_+ + A_4^{[2]}$ and $y_- \in By_+ + Bx_+ + A_4^{[2]}$. As a consequence we get the following.

Proposition 5.8. $A_4 = A_4^{[3]} = A_4^{[2]} + Bx_+ + Bx_- + By_+ = A_4^{[2]} + Bx_+ + Bx_- + By_-$.

For $x \in A_4^\times$, we let $[x]$ denote its class in $B^\times\backslash A_4^\times/B^\times$, and we write $x \sim y$ for $[x] = [y]$.

Lemma 5.9. *Let $E_2 = \{s_2^\alpha s_1^\beta s_3^\gamma s_2^\delta \mid \alpha,\beta,\gamma,\delta \in \{0,1,-1\}\} \subset A_4^\times$. The image of $[E_2]$ of E_2 in $B^\times\backslash A_4^\times/B^\times$ has cardinality at most 13, and is equal to $\mathcal{S}_2$, with*

$$\begin{aligned}\mathcal{S}_2 = \{&[1], [s_2], [s_2^{-1}], [s_2s_1^{-1}s_2], [s_2s_3^{-1}s_2], [s_2s_1s_3s_2], [s_2s_1^{-1}s_3s_2], [s_2s_1^{-1}s_3^{-1}s_2], [s_2s_1s_3^{-1}s_2^{-1}],\\ &[s_2s_1^{-1}s_3s_2^{-1}], [s_2s_1s_3^{-1}s_2^{-1}], [s_2^{-1}s_1s_3s_2^{-1}], [s_2^{-1}s_1^{-1}s_3^{-1}s_2^{-1}]\}\end{aligned}$$

Proof. Clearly $\mathcal{S}_2 \subset [E_2]$, hence we only need to prove $[E_2] \subset \mathcal{S}_2$. In view of $s_2^\alpha s_i^\alpha s_2^\alpha = s_i^\alpha s_2^\alpha s_i^\alpha$, $s_2^{-1}s_i^\alpha s_2 = s_is_2^\alpha s_i^{-1}$, $s_2s_i^\alpha s_2^{-1} = s_i^{-1}s_2^\alpha s_i$ for $\alpha \in \{-1,1\}$ and $i \in \{1,3\}$, we have $[s_2^\alpha s_i^\beta s_2^\gamma] \in \mathcal{S}_2$ for all α,β,γ. Among the $s_2s_1^\alpha s_3^\beta s_2$ for $\alpha,\beta \in \{-1,1\}$, we have $[s_2s_1^\alpha s_3^\beta s_2] \in \mathcal{S}_2$ because $s_2s_1^\alpha s_3^\beta s_2 \sim s_2s_1^{-1}s_3s_2$: indeed, we have the identity $s_2s_1^{-1}s_3s_2 = s_1^{-1}s_3(s_2s_1s_3^{-1}s_2)s_1^{-1}s_3$ in the braid group on 4 strands (because $s_1^{-1}s_3s_2s_1s_3^{-1}s_2s_1^{-1}s_3 = s_1^{-1}(s_3s_2s_3^{-1})s_1s_2s_1^{-1}s_3 = s_1^{-1}s_2^{-1}s_3(s_2s_1s_2)s_1^{-1}s_3 = s_1^{-1}s_2^{-1}s_3s_1s_2s_1s_1^{-1}s_3 = s_1^{-1}s_2^{-1}s_1(s_3s_2s_3) = (s_1^{-1}s_2^{-1}s_1)s_2s_3s_2 = s_2s_1^{-1}s_2^{-1}s_2s_3s_2 = s_2s_1^{-1}s_3s_2$). Among the $s_2s_1^\alpha s_3^\beta s_2^{-1}$ for $\alpha,\beta \in \{-1,1\}$, we have $[s_2s_1^\alpha s_3^\beta s_2^{-1}] \in \mathcal{S}_2$ because $s_2s_1s_3s_2^{-1} \sim s_2s_1s_3^{-1}s_2$: indeed, we have $s_1(s_2s_1s_3s_2^{-1})s_3^{-1} = (s_1s_2s_1)s_3s_2^{-1}s_3^{-1} = s_2s_1(s_2s_3s_2^{-1})s_3^{-1} = s_2s_1s_3^{-1}s_2s_3s_3^{-1} = s_2s_1s_3^{-1}s_2$.

Again for $\alpha, \beta \in \{-1, 1\}$, we have $[s_2^{-1}s_1^{\alpha}s_3^{\beta}s_2] \in \mathcal{S}_2$ because of the following identities

$$(1)\ s_2^{-1}s_1^{-1}s_3s_2 \sim s_2s_1s_3^{-1}s_2^{-1} \quad (2)\ s_2^{-1}s_1s_3s_2 \sim s_2s_1s_3^{-1}s_2$$
$$(3)\ s_2^{-1}s_1s_3^{-1}s_2 \sim s_2s_1^{-1}s_3s_2^{-1} \quad (4)\ s_2^{-1}s_1^{-1}s_3^{-1}s_2 \sim s_2s_1^{-1}s_3^{-1}s_2^{-1}$$

We prove these identities now. We have $s_3(s_2s_3^{-1}s_1s_2^{-1})s_1^{-1} = (s_3s_2s_3^{-1})s_1s_2^{-1}s_1^{-1} = s_2^{-1}s_3(s_2s_1s_2^{-1})s_1^{-1} = s_2^{-1}s_3s_1^{-1}s_2s_1s_1^{-1} = s_2^{-1}s_3s_1^{-1}s_2$ hence $s_2s_3^{-1}s_1s_2^{-1} \sim s_2^{-1}s_3s_1^{-1}s_2$ that is (1). By applying Ad Δ this implies $s_2s_1^{-1}s_3s_2^{-1} \sim s_2^{-1}s_1s_3^{-1}s_2$ that is (3). We have $s_3^{-1}(s_2^{-1}s_1s_3s_2)s_1 = (s_3^{-1}s_2^{-1}s_3)s_1s_2s_1 = s_2s_3^{-1}(s_2^{-1}s_1s_2)s_1 = s_2s_3^{-1}s_1s_2s_1^{-1}s_1 = s_2s_3^{-1}s_1s_2$ hence $s_2^{-1}s_1s_3s_2 \sim s_2s_3^{-1}s_1s_2$ that is (2).

We have $s_1(s_2^{-1}s_1^{-1}s_3s_2^{-1})s_3^{-1} = (s_1s_2^{-1}s_1^{-1})s_3s_2^{-1}s_3^{-1} = s_2^{-1}s_1^{-1}(s_2s_3s_2^{-1})s_3^{-1} = s_2^{-1}s_1^{-1}s_3^{-1}s_2s_3s_3^{-1} = s_2^{-1}s_1^{-1}s_3^{-1}s_2$ hence $s_2^{-1}s_1^{-1}s_3s_2^{-1} \sim s_2^{-1}s_1^{-1}s_3^{-1}s_2$. Moreover, we have $s_1(s_2s_1^{-1}s_3^{-1}s_2^{-1})s_3^{-1} = s_1s_2s_1^{-1}(s_3^{-1}s_2^{-1}s_3^{-1}) = s_1(s_2s_1^{-1}s_2^{-1})s_3^{-1}s_2^{-1} = s_1s_1^{-1}s_2^{-1}s_1s_3^{-1}s_2^{-1} = s_2^{-1}s_1s_3^{-1}s_2^{-1}$ hence $s_2s_1^{-1}s_3^{-1}s_2^{-1} \sim s_2^{-1}s_1s_3^{-1}s_2^{-1}$. Applying Δ we get $s_2s_1^{-1}s_3^{-1}s_2^{-1} \sim s_2^{-1}s_3s_1^{-1}s_2^{-1}$, hence $s_2s_1^{-1}s_3^{-1}s_2^{-1} \sim s_2^{-1}s_3s_1^{-1}s_2^{-1} \sim s_2^{-1}s_1^{-1}s_3^{-1}s_2$ hence (4).

Now, for $\alpha, \beta \in \{-1, 1\}$, we have $[s_2^{-1}s_1^{\alpha}s_3^{\beta}s_2^{-1}] \in \mathcal{S}_2$ because $s_2^{-1}s_1s_3^{-1}s_2^{-1} \sim s_2s_1^{-1}s_3^{-1}s_2^{-1}$ and $s_2^{-1}s_1^{-1}s_3s_2^{-1} \sim s_2s_1^{-1}s_3^{-1}s_2^{-1}$ as we proved above, and this concludes the proof. □

From this we get $A_4 = \sum_{\sigma\in[E_2]} B\sigma B + Bx_+ + Bx_- + By_- = \sum_{\sigma\in\mathcal{S}_2} B\sigma B + Bx_+ + Bx_- + By_-$.
We write $\mathcal{S}_2 = \mathcal{S}_2^1 \cup \mathcal{S}_2^{\Delta} \cup \mathcal{S}_2^{\alpha} \cup \mathcal{S}_2^{\beta} \cup \mathcal{S}_2^0$ with

$$\mathcal{S}_2^1 = \{[1], [s_2], [s_2^{-1}]\} \qquad \mathcal{S}_2^{\Delta} = \{[s_2s_1s_3s_2], [s_2^{-1}s_1^{-1}s_3^{-1}s_2^{-1}]\}$$
$$\mathcal{S}_2^{\alpha} = \{[s_2s_1^{-1}s_2], [s_2s_3^{-1}s_2]\} \qquad \mathcal{S}_2^{\beta} = \{[s_2s_1s_3^{-1}s_2^{-1}], [s_2s_1^{-1}s_3s_2^{-1}]\}$$
$$\mathcal{S}_2^0 = \{[s_2s_1^{-1}s_3s_2], [s_2s_1^{-1}s_3^{-1}s_2], [s_2s_1^{-1}s_3^{-1}s_2^{-1}], [s_2^{-1}s_1s_3s_2^{-1}]\}$$

Recall that $B = u_1u_3 = u_3u_1$. We prove the following.

Lemma 5.10. (1) $s_2s_1s_3s_2B \subset Bs_2s_1s_3s_2$, $s_2^{-1}s_1^{-1}s_3^{-1}s_2^{-1}B \subset Bs_2^{-1}s_1^{-1}s_3^{-1}s_2^{-1}$
(2) $s_2s_1^{-1}s_2B \subset Bs_2s_1^{-1}s_2u_3 + A_4^{[1]}$, $s_2s_3^{-1}s_2B \subset Bs_2s_3^{-1}s_2u_1 + A_4^{[1]}$
(3) $s_2s_1s_3^{-1}s_2^{-1}B \subset Bs_2s_1s_3^{-1}s_2^{-1}u_1$, $s_2s_1^{-1}s_3s_2^{-1}B \subset Bs_2s_1^{-1}s_3s_2^{-1}u_3$.

Proof. We have $\Delta = s_1s_3(s_2s_1s_3s_2) = (s_2s_1s_3s_2)s_1s_3$ and $\Delta B = B\Delta$ hence $s_2s_1s_3s_2B = (s_1s_3)^{-1}\Delta B = B(s_1s_3)^{-1}\Delta = B(s_2s_1s_3s_2)$. Applying Φ (or considering Δ^{-1}) we get $s_2^{-1}s_1^{-1}s_3^{-1}s_2^{-1}B = Bs_2^{-1}s_1^{-1}s_3^{-1}s_2^{-1}$ hence (1).

By Lemma 3.5 we have $(s_2s_1^{-1}s_2)s_1^{-1} \in s_1^{-1}(s_2s_1^{-1}s_2) + A_4^{[1]}$ hence $(s_2s_1^{-1}s_2)u_1 \subset u_1(s_2s_1^{-1}s_2) + A_4^{[1]}$ and $(s_2s_1^{-1}s_2)u_1 \subset B(s_2s_1^{-1}s_2) + A_4^{[1]}$. Since $B = u_1u_3$ this yields $(s_2s_1^{-1}s_2)B = (s_2s_1^{-1}s_2)u_1u_3 \subset B(s_2s_1^{-1}s_2)u_3 + A_4^{[1]}$. Using Ad Δ this implies $(s_2s_3^{-1}s_2)B \subset B(s_2s_3^{-1}s_2)u_1 + A_4^{[1]}$, hence (2). Finally, $(s_2s_1s_3^{-1}s_2^{-1})s_3 = s_2s_1(s_3^{-1}s_2^{-1}s_3) = (s_2s_1s_2)s_3^{-1}s_2^{-1} = s_1(s_2s_1s_3^{-1}s_2^{-1})$ hence $(s_2s_1s_3^{-1}s_2^{-1})u_3 \subset B(s_2s_1s_3^{-1}s_2^{-1})$ whence, using $B = u_3u_1$, $(s_2s_1s_3^{-1}s_2^{-1})B \subset B(s_2s_1s_3^{-1}s_2^{-1})u_1$ and, applying Ad Δ, $(s_2s_1^{-1}s_3s_2^{-1})B \subset B(s_2s_1^{-1}s_3s_2^{-1})u_3$, which proves (3). □

Proposition 5.11. (1) $A_4^{[1]} = B + Bs_2B + Bs_2^{-1}B$ *is equal to*

$$B + \sum_{a,b\in\{0,1,-1\}} Bs_2s_1^as_3^b + \sum_{a,b\in\{0,1,-1\}} Bs_2^{-1}s_1^as_3^b$$

(2) $A_4^{[2]} = Bu_2A_4^{[1]} = A_4^{[1]}u_2B$ *is equal to*

$$A_4^{[1]} + \sum_{x\in\mathcal{S}_2^{\Delta}} Bx + \sum_{a\in\{0,1,-1\}} Bs_2s_1^{-1}s_2s_3^a + \sum_{a\in\{0,1,-1\}} Bs_2s_3^{-1}s_2s_1^a + \sum_{a\in\{0,1,-1\}} Bs_2s_1s_3^{-1}s_2^{-1}s_1^a$$
$$+ \sum_{a\in\{0,1,-1\}} Bs_2s_1^{-1}s_3s_2^{-1}s_3^a + \sum_{\substack{x\in\mathcal{S}_2^0\\ a,b\in\{0,1,-1\}}} Bxs_1^as_3^b$$

(3) $A_4 = A_4^{[3]} = A_4^{[2]} + Bx_+ + Bx_- + By_-$

Proof. (1) is clear, (3) has been proved before, and (2) is an immediate consequence of $A_4^{[2]} = A_4^{[1]} + \sum_{x\in\mathcal{S}_2} BxB$ and of Lemma 5.10. □

Corollary 5.12. *As a B-module, A_4 is generated by 72 elements, which are images of elements of the braid group on 4 strands.*

Proof. By Proposition 5.11, $A_4^{[1]}$ is generated by $1 + 9 + 9 = 19$ elements, $A_4^{[2]}$ by $A_4^{[1]}$ and $|\mathcal{S}_2^{\Delta}| + 4 \times 3 + 9 \times |\mathcal{S}_2^0| \leq 2 + 12 + 9 \times 4 = 50$ elements, and $A_4^{[3]}$ by $A_4^{[2]}$ and 3 elements. Thus $A_4 = A_4^{[3]}$ is generated by 72 elements, all originating from the braid group. □

6. The algebra A_5

Recall $w^+ = s_3 s_2^{-1} s_1 s_2^{-1} s_3$, $w^- = s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} \in A_4$. Our first goal is to prove the following.

Theorem 6.1.

$$\begin{aligned} A_5 = {} & A_4 + A_4 s_4 A_4 + A_4 s_4^{-1} A_4 + A_4 s_4 s_3^{-1} s_4 A_4 + A_4 s_4^{-1} s_3 s_2^{-1} s_3 s_4^{-1} A_4 + A_4 s_4 s_3^{-1} s_2 s_3^{-1} s_4 A_4 \\ & + A_4 s_4^{-1} w^+ s_4^{-1} A_4 + A_4 s_4 w^- s_4 A_4 + A_4 s_4^{-1} w^- s_4^{-1} A_4 + A_4 s_4 w^+ s_4 A_4 + A_4 s_4 w^- s_4 w^- s_4 A_4 \\ & + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 \end{aligned}$$

We denote again U the right-hand side. We let $A_5^{(0)} = A_4$ and $A_5^{(n+1)} = A_5^{(n)} u_4 A_4$. This defines an increasing sequence of A_4 sub-bimodules of A_5. An immediate consequence of Theorem 4.1 is $sh(A_4) \subset U$. Also, we have $u_4 \subset U$ hence $A_5^{(1)} = A_4 u_4 A_4 \subset U$.

Lemma 6.2. *$u_4 A_4 u_4 \subset U$, hence $A_5^{(2)} = A_4 u_4 A_4 u_4 A_4 \subset U$.*

Proof. According to Theorem 4.1, we have $A_4 = A_3 + A_3 s_3 A_3 + A_3 s_3^{-1} A_3 + A_3 s_3 s_2^{-1} s_3 A_3 + A_3 w^- + A_3 w^+$, hence $u_4 A_4 u_4 \subset A_3 u_4 + A_4 u_4 u_3 u_4 A_4 + A_4 u_4 s_3 s_2^{-1} s_3 u_4 A_3 + A_3 u_4 w^- u_4 + A_3 u_4 w^+ u_4$. We have $A_3 u_4 + A_4 u_4 u_3 u_4 A_4 + A_4 u_4 s_3 s_2^{-1} s_3 u_4 A_3 \subset A_4 sh(A_4) A_4 \subset A_4 U A_4 \subset U$. Moreover, since by definition $s_4^{\alpha} w^{\beta} s_4^{\alpha} \in U$ for all $\alpha, \beta \in \{-1, 1\}$, we have $s_4 A_4 s_4 \subset U$, $s_4^{-1} A_4 s_4^{-1} \subset U$, and we only need to prove $s_4 w^{\pm} s_4^{-1} \in U$ and $s_4^{-1} w^{\pm} s_4 \in U$. We have $w^{\pm} \in s_3^{\alpha} A_3 s_3^{\alpha}$ for some $\alpha \in \{-1, 1\}$, hence $s_4^{\alpha} w^{\pm} s_4^{-\alpha} \in s_4^{\alpha} s_3^{\alpha} A_3 s_3^{\alpha} s_4^{-\alpha} = s_3^{-\alpha}(s_3^{\alpha} s_4^{\alpha} s_3^{\alpha}) A_3 s_3^{\alpha} s_4^{-\alpha} = s_3^{-\alpha} s_4^{\alpha} s_3^{\alpha} s_4^{\alpha} A_3 s_3^{\alpha} s_4^{-\alpha} \subset A_4 s_4^{\alpha} s_3^{\alpha} A_3 (s_4^{\alpha} s_3^{\alpha} s_4^{-\alpha}) \subset A_4 s_4^{\alpha} s_3^{\alpha} A_3 s_3^{-\alpha} s_4^{\alpha} s_3^{\alpha}$ by Lemma 2.1. Now $A_4 s_4^{\alpha} s_3^{\alpha} A_3 s_3^{-\alpha} s_4^{\alpha} s_3^{\alpha}$ lies inside $A_4 (s_4^{\alpha} A_4 s_4^{\alpha}) A_4 \subset A_4 U A_4 \subset U$, as we already proved. $\square$

6.1. The A_4-bimodule $A_5^{(3)}/A_5^{(2)}$: first reduction.

Proposition 6.3. *If $p \le 5$, $q \le 5$ and $(p, q) \ne (5, 5)$, then for all $x \in u_4 u_{i_1} \dots u_{i_p} u_4 u_{j_1} \dots u_{j_q} u_4$ we have $x \in A_4 u_4 A_4 u_4 A_4$, for all choices of $i_1, \dots, i_p, j_1, \dots, j_q \in \{1, 2, 3\}$, unless $(p, q) \in \{(5, 4), (4, 5)\}$ and $x \in s_4 u_3 u_2 u_1 u_3 u_2 s_4 u_1 u_3 u_2 u_3 s_4 \cup s_4^{-1} u_3 u_2 u_1 u_3 u_2 s_4^{-1} u_1 u_3 u_2 u_3 s_4^{-1}$.*

Proof. Note that $sh(A_4) \subset A_4 u_4 A_4 u_4 A_4$. By application of Ψ we may assume $p \ge q \ge 1$. We prove the statement by induction on (p, q), using lexicographic ordering. By commutation relations we can assume $i_1 \notin \{1, 2\}$ hence $i_1 = 3$, and similarly $j_q = 3$. In case $(p, q) = (1, 1)$ we have then $u_4 u_3 u_4 u_3 u_4 \subset sh(A_4) \subset A_4 u_4 A_4 u_4 A_4$. More generally, in the cases (1, 1), (2, 1), (2, 2), (3, 1), using only commutation relations we check that the corresponding algebras are necessarily included in $A_4 sh(A_4) A_4 \subset A_4 u_4 A_4 u_4 A_4$.

If $(p, q) = (3, 2)$, the only case which is not clearly included in $A_4 sh(A_4) A_4$ is $u_4 u_3 u_2 u_1 u_4 u_2 u_3 u_4 = u_4 u_3 u_4 u_2 u_1 u_2 u_3 u_4$, and we have $u_4 u_3 u_4 \subset u_3 s_4 s_3^{-1} s_4 + u_3 u_4 u_3$ by Theorem 3.2 hence $u_4 u_3 u_4 u_2 u_1 u_2 u_3 u_4 \subset u_3 s_4 s_3^{-1} s_4 u_2 u_1 u_2 u_3 u_4 + u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 \subset u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4 u_3 u_4 + A_4 u_4 A_4 u_4 A_4$. Again $s_4 u_3 u_4 \subset s_4^{-1} s_3 s_4^{-1} u_3 + u_3 u_4 u_3$ by Theorem 3.2 hence $u_3 s_4 s_3^{-1} u_2 u_1 u_2 (s_4 u_3 u_4) \subset u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4^{-1} s_3 s_4^{-1} + A_4 u_4 A_4 u_4 A_4$ and $u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4^{-1} s_3 s_4^{-1} = u_3 (s_4 s_3^{-1} s_4^{-1}) u_2 u_1 u_2 s_3 s_4^{-1} = u_3 s_3^{-1} s_4^{-1} s_3 u_2 u_1 u_2 s_3 s_4^{-1} \subset A_4 u_4 A_4 u_4 A_4$.

If $(p, q) = (3, 3)$, the corresponding algebra is either included in $A_4 sh(A_4) A_4 \subset U$, or can be reduced using commutation relations to the case (3, 2), or we are dealing with the remaining case $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 u_4$ (or its image under the natural anti-isomorphism $u_4 u_3 u_2 u_3 u_4 u_1 u_2 u_3 u_4$). We want to prove $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 u_4 \subset A_4 u_4 A_4 u_4 A_4$. Up to using the natural isomorphism induced by $s_i \mapsto s_i^{-1}$, we only need to prove $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 s_4 \subset A_4 u_4 A_4 u_4 A_4$. We have $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 s_4 = u_4 u_3 u_4 u_2 u_1 u_3 u_2 u_3 s_4$ and we know from Theorem 3.2 that $u_4 u_3 u_4 \subset A_4 s_4^{-1} s_3 s_4^{-1} + u_3 u_4 u_3$, hence $u_4 u_3 u_4 u_2 u_1 u_3 u_2 u_3 s_4 \subset A_4 s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 + A_4 u_4 A_4 u_4 A_4$. Now, using $u_3 u_2 u_3 \subset s_3 s_2^{-1} s_3 u_2 + u_2 u_3 u_3$ we have $s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 \subset s_4^{-1} s_3 s_4^{-1} u_2 u_1 s_3 s_2^{-1} s_3 u_2 s_4 + s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_2 u_3 u_2 s_4$. But $s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_2 u_3 u_2 s_4 = s_4^{-1} s_3 u_2 u_1 s_4^{-1} u_2 u_3 s_4 u_2 \subset A_4 u_4 A_4 u_4 A_4$ by the induction assumption, hence $s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 \subset s_4^{-1} s_3 s_4^{-1} u_2 u_1 s_3 s_2^{-1} s_3 s_4 u_2 + A_4 u_4 A_4 u_4 A_4$. Now we need to prove $s_4^{-1} s_3 s_4^{-1} s_2^{\alpha} s_1^{\beta} s_3 s_2^{-1} s_3 s_4 \in A_4 u_4 A_4 u_4 A_4$ for $\alpha, \beta \in \{-1, 1\}$. If $\alpha = 1$, this holds true because $s_4^{-1} s_3 s_4^{-1} s_2 u_1 s_3 s_2^{-1} s_3 s_4 = s_4^{-1} s_3 s_2 u_1 s_3 s_4 s_3^{-1} s_2^{-1} s_3 s_4 s_3^{-1} = s_2 s_4^{-1} s_3 s_2 u_1 s_4 s_2 s_3^{-1} s_4 s_2^{-1} s_3^{-1} \subset A_4 u_4 A_4 u_4 A_4$ by the induction assumption. We thus assume $\alpha = -1$. If $\beta = 1$, then

$$\begin{aligned} s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 (s_3 s_2^{-1} s_3) s_4 & \subset s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 s_3^{-1} s_2 s_3^{-1} s_4 u_2 && + A_4 u_4 A_4 u_4 A_4 \text{ (Lemmas 2.4 + 2.3)} \\ & \subset s_1 s_1^{-1} s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 s_3^{-1} s_2 s_3^{-1} s_4 u_2 && + A_4 u_4 A_4 u_4 A_4 \\ & \subset s_1 s_4^{-1} s_3 s_4^{-1} (s_1^{-1} s_2^{-1} s_1) s_3^{-1} s_2 s_3^{-1} s_4 u_2 && + A_4 u_4 A_4 u_4 A_4 \\ & \subset s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2^{-1} s_3^{-1} s_2 s_3^{-1} s_4 u_2 && + A_4 u_4 A_4 u_4 A_4 \\ & \subset s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} sh(A_3) s_4 u_2 && + A_4 u_4 A_4 u_4 A_4 \\ & \subset s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_3 s_2^{-1} s_3 u_2 s_4 u_2 && + A_4 u_4 A_4 u_4 A_4 \end{aligned}$$

by Theorem 3.2 and the induction assumption, and we already proved $s_4^{-1}s_3s_4^{-1}s_2s_1^{-1}s_3s_2^{-1}s_3u_2s_4 = s_4^{-1}s_3s_4^{-1}s_2s_1^{-1}s_3s_2^{-1}s_3s_4u_2 \subset A_4u_4A_4u_4A_4$. The remaining case is then $(\alpha, \beta) = (-1, -1)$, for which we have

$$\begin{aligned}
s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_4 &\subset s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1^{-1}s_3^{-1}s_2s_3^{-1}s_4A_4 && + A_4u_4A_4u_4A_4 \text{ (Lemma 2.4)}\\
&\subset s_3s_3^{-1}(s_4^{-1}s_3s_4^{-1})s_2^{-1}s_3^{-1}s_1^{-1}s_2s_3^{-1}s_4A_4 && + A_4u_4A_4u_4A_4\\
&\subset s_3(s_4^{-1}s_3s_4^{-1})s_3^{-1}s_2^{-1}s_3^{-1}s_1^{-1}s_2s_3^{-1}s_4A_4 && + A_4u_4A_4u_4A_4 \text{ (Lemma 2.3)}\\
&\subset A_4s_4^{-1}s_3s_4^{-1}(s_3^{-1}s_2^{-1}s_3^{-1})s_1^{-1}s_2s_3^{-1}s_4A_4 && + A_4u_4A_4u_4A_4\\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_3^{-1}(s_2^{-1}s_1^{-1}s_2)s_3^{-1}s_4A_4 && + A_4u_4A_4u_4A_4\\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_3^{-1}s_1s_2^{-1}s_1^{-1}s_3^{-1}s_4A_4 && + A_4u_4A_4u_4A_4\\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1(s_3^{-1}s_2^{-1}s_3^{-1})s_4s_1^{-1}A_4 && + A_4u_4A_4u_4A_4\\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2^{-1}s_4s_1^{-1}A_4 && + A_4u_4A_4u_4A_4\\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1s_2^{-1}s_3^{-1}s_4s_2^{-1}A_4 && + A_4u_4A_4u_4A_4\\
&\subset A_4s_4^{-1}s_3s_2^{-1}s_1s_4^{-1}s_2^{-1}s_3^{-1}s_4A_4 && + A_4u_4A_4u_4A_4\\
&\subset A_4u_4A_4u_4A_4 && \text{(induction assumption)}
\end{aligned}$$

and this concludes the case $(p, q) = (3, 3)$.

All cases $(4, q)$ for $q = 1, 2, 4$ can be easily reduced to smaller cases by using commutation relations and relations $u_iu_ju_iu_j = u_ju_iu_ju_i$. Most cases for $(4, 3)$ can also be reduced this way, except for one remaining case $u_4u_3u_2u_3u_1u_4u_3u_2u_3u_4$. Using Φ, we only need to prove $u_4u_3u_2u_3u_1s_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4$. Using the induction assumption and Theorem 3.2 on $sh(A_3)$, we get

$$\begin{aligned}
u_4(u_3u_2u_3)u_1s_4(u_3u_2u_3)u_4 &\subset u_4u_2s_3s_2^{-1}s_3u_1s_4s_3s_2^{-1}s_3u_2u_4 && + A_4u_4A_4u_4A_4\\
&\subset A_3u_4s_3s_2^{-1}u_1(s_3s_4s_3)s_2^{-1}s_3u_4A_3 && + A_4u_4A_4u_4A_4\\
&\subset A_3u_4s_3s_2^{-1}u_1s_4s_3s_4s_2^{-1}s_3u_4A_3 && + A_4u_4A_4u_4A_4\\
&\subset A_3(u_4s_3s_4)s_2^{-1}u_1s_3s_2^{-1}(s_4s_3u_4)A_3 && + A_4u_4A_4u_4A_4\\
&\subset A_3u_3u_4u_3s_2^{-1}u_1s_3s_2^{-1}u_3u_4u_3A_3 && + A_4u_4A_4u_4A_4
\end{aligned}$$

by Lemma 2.1, which proves the claim.

We now deal with the cases $(5, q)$ with $1 \leq q < 5$. We can assume that $u_{i_1}\dots u_{i_p} = u_3u_2u_1u_2u_3$ or $u_{i_1}\dots u_{i_p} = u_3u_2u_1u_3u_2$, because otherwise we can reduce to smaller cases by using commutation relations and the relation $u_au_bu_au_b = u_bu_au_bu_a$. From this remark one easily checks that the cases $(5, 1)$ are readily reduced to smaller cases, and also the cases $(5, 2)$ except for the case $u_4u_3u_2u_1u_2u_3u_4u_2u_3u_4 = u_4u_3u_2u_1u_2u_3u_2u_4u_3u_4$ that we tackle now: we have $u_3u_2u_1u_2u_3u_2 \subset A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ by Theorem 4.1, hence

$$\begin{aligned}
u_4u_3u_2u_1u_2u_3u_2u_4u_3u_4 &\subset u_4A_3u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_1u_2u_3u_4u_3u_4\\
&\subset A_3u_4u_3u_4A_3u_3u_4 + A_3u_4u_3u_2u_3u_4A_3u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4\\
&\subset A_3u_4u_3u_4u_2u_1u_2u_1u_3u_4 + A_3u_4u_3u_2u_3u_4u_2u_1u_2u_1u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4\\
&\subset A_3u_4u_3u_4u_2u_1u_2u_3u_4A_2 + A_3u_4u_3u_2u_3u_4u_2u_1u_2u_3u_4A_2 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4
\end{aligned}$$

using $A_3 = u_2u_1u_2u_1$, and we are thus reduced to smaller cases.

When $(p, q) = (5, 3)$, the only nontrivial case (up to commutation and $u_au_bu_au_b = u_bu_au_bu_a$ relations) is $u_4u_3u_2u_1u_2u_3u_4u_3u_2u_3u_4$. We have $u_2u_3u_4u_3u_2u_3u_4 \subset sh(A_4) \subset A_4u_4A_4 + sh(A_3)u_4u_3u_4A_4 + u_4u_3u_2u_3u_4A_4$ by Theorem 4.1, hence

$$u_4u_3u_2u_1u_2u_3u_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4 + u_4u_3u_2u_1sh(A_3)u_4u_3u_4A_4 + u_4u_3u_2u_1u_4u_3u_2u_3u_4A_4$$

and we have $u_4u_3u_2u_1u_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4$ by the induction assumption, and, since $sh(A_3) = u_2u_3u_2u_3$ by Theorem 3.2, $u_4u_3u_2u_1sh(A_3)u_4u_3u_4$ is a subset of

$$u_4u_3u_2u_1u_2u_3u_2(u_3u_4u_3u_4) = u_4u_3u_2u_1u_2u_3u_2u_4u_3u_4u_3 = u_4u_3u_2u_1u_2u_3u_4u_2u_3u_4u_3$$

and we are reduced to case $(5, 2)$.

When $(p, q) = (5, 4)$, the only nontrivial cases are $u_4u_3u_2u_1u_2u_3u_4u_2u_1u_2u_3u_4$ and $u_4u_3u_2u_1u_3u_2u_4u_3u_1u_2u_3u_4$. In the first case, $u_4u_3u_2u_1u_2u_3u_4u_2u_1u_2u_3u_4 = u_4u_3u_2u_1u_2u_3u_2u_1u_2u_4u_3u_4 \subset u_4A_4u_4u_3u_4$. By Theorem 4.1, we have $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ hence

$$\begin{aligned}
u_4A_4u_4u_3u_4 &\subset u_4A_3u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_1u_2u_3u_4u_3u_4\\
&\subset A_3u_4u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_3u_4A_3u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4\\
&\subset A_4u_4A_4u_4A_4
\end{aligned}$$

by the induction assumption and $A_3 = u_2u_1u_2u_1$.

In the second case, we need to consider the sets $s_4^{\alpha}u_3u_2u_1u_3u_2s_4^{\beta}u_3u_1u_2u_3s_4^{\gamma}$ with $\alpha, \beta, \gamma \in \{-1, 1\}$, and we can assume that two of them have distinct signs, otherwise we are in the exceptional case of the statement. Up to using Φ and Ψ, we can assume $\gamma = 1$ and $\beta = -1$. We are thus considering expressions of the type $u_4u_3u_2u_1u_3u_2s_4^{-1}u_3u_1u_2u_3s_4 = u_4u_3u_2u_3u_1u_2u_1s_4^{-1}u_3u_2u_3s_4$. Notice that

$$\begin{aligned} & u_4u_3u_2u_3(u_2u_1u_2)s_4^{-1}u_3u_2u_3s_4 = u_4(u_3u_2u_3u_2)u_1u_2s_4^{-1}u_3u_2u_3s_4 \\ = {} & u_4u_2u_3u_2u_3u_1u_2s_4^{-1}u_3u_2u_3s_4 = u_2u_4u_3u_2u_3u_1u_2s_4^{-1}u_3u_2u_3s_4 \end{aligned}$$

hence reduces to smaller cases. As a consequence, among the natural spanning set of $u_1u_2u_1$, only the $s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}$ do not reduce to smaller cases, and so we may restrict ourselves to these. Moreover, using $u_3u_2u_3 \subset u_2s_3^{-1}s_2s_3^{-1} + u_2u_3u_2$ and $u_3u_2u_3 \subset s_3s_2^{-1}s_3u_2 + u_2u_3u_2$ we are reduced to expressions of the form $u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_4^{-1}s_3s_2^{-1}s_3s_4$. We then have $u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_4^{-1}s_3s_2^{-1}s_3s_4 = u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_3s_4s_2s_3^{-1}s_4s_2^{-1}s_3^{-1}$ so we now need to prove that $u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_3s_4s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$. When $\alpha = 1$ we get $u_4s_3^{-1}s_2s_3^{-1}s_1s_2^{-1}s_1s_3s_4s_2s_3^{-1}s_4 = s_1u_4s_3^{-1}s_2s_1s_3^{-1}s_2^{-1}s_4s_1s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ by the induction assumption. When $\alpha = -1$ we get $u_4s_3^{-1}s_2s_3^{-1}s_1^{-1}s_2s_1^{-1}s_3s_4s_2s_3^{-1}s_4 = u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_1s_2^{-1}s_3^{-1}s_4s_1^{-1} \subset A_4u_4A_4u_4A_4$ by the induction assumption.

This concludes the case (5, 4) and the proof of the proposition. □

Lemma 6.4. $u_4u_3u_2u_3u_1u_2u_1u_4u_3u_2u_3u_4 \subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)u_2 + A_4u_4A_4u_4A_4$.

Proof. By Proposition 6.3 it is enough to prove that $s_4u_3u_2u_3u_1u_2u_1s_4u_3u_2u_3s_4$ is a subset of $A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4u_2 + A_4u_4A_4u_4A_4$ and, as noted in the proof of Proposition 6.3, we can restrict to the forms $s_4u_3u_2u_3s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_4u_3u_2u_3s_4$. Moreover, since $s_1s_2^{-1}s_1 = s_2^{-1}s_1^{-1}s_2s_1^2 \in s_2^{-1}s_1^{-1}s_2u_1$, and $s_1^{-1}s_2u_1 \subset Rs_1^{-1}s_2s_1^{-1} + u_2u_1u_2$, we get

$$\begin{aligned} s_4u_3u_2u_3s_1s_2^{-1}s_1s_4u_3u_2u_3s_4 & \subset s_4u_3u_2u_3s_2^{-1}s_1^{-1}s_2u_1s_4u_3u_2u_3s_4 \subset s_4(u_3u_2u_3u_2)s_1^{-1}s_2u_1s_4u_3u_2u_3s_4 \\ & \subset s_4(u_2u_3u_2u_3)s_1^{-1}s_2u_1s_4u_3u_2u_3s_4 \subset u_2s_4u_3u_2u_3s_1^{-1}s_2u_1s_4u_3u_2u_3s_4 \\ & \subset u_2s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4(u_3u_2u_3u_2)u_1u_2s_4u_3u_2u_3s_4 \\ & \subset u_2s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4u_2u_3u_2u_3u_1u_2s_4u_3u_2u_3s_4 \\ & \subset u_2s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4u_3u_2u_3u_1u_2s_4u_3u_2u_3s_4 \\ & \subset u_2s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 + A_4u_4A_4u_4A_4 \end{aligned}$$

by Proposition 6.3. We can thus restrict to $s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4$. Moreover, using that $u_3u_2u_3 \subset u_2s_3s_2^{-1}s_3 + u_2u_3u_2$ and $u_3u_2u_3 \subset s_3^{-1}s_2s_3^{-1}u_2 + u_2u_3u_2$ leads to $s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 \subset u_2s_4s_3s_2^{-1}s_3s_1^{-1}s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4u_2 + A_4u_4A_4u_4A_4$ by Proposition 6.3. Now $s_4s_3s_2^{-1}s_3s_1^{-1}s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 = s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4$ belongs to $A_4s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4$ and this proves the claim. □

Lemma 6.5. $u_4u_3u_2u_1u_2u_3u_4u_2u_3u_1u_2u_3u_4 \subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 + A_4u_4A_4u_4A_4$.

Proof. We consider the expression $u_4s_3^{\alpha}u_2u_1u_2s_3^{\beta}u_4u_2u_3u_1u_2u_3u_4$ and we first assume $\alpha = \beta$; by applying if necessary Φ, we can then assume $\alpha = \beta = -1$. Since $u_2u_1u_2 \subset u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ we have

$$\begin{aligned} u_4s_3^{-1}u_2u_1u_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 & \subset u_4s_3^{-1}u_1s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 + u_4s_3^{-1}u_1u_2u_1s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 \\ & \subset u_1u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 + u_1u_4s_3^{-1}u_2u_1s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 \end{aligned}$$

and we are reduced to $u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4$ by Lemmas 6.3 and 6.4. Now we have that $u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4$ is equal to $u_4(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_2u_1u_4u_3u_2u_3u_4$ and that $(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_2u_1 \subset A_3(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}) + A_3u_3A_3 + A_3s_3s_2^{-1}s_3A_3$ by Lemma 4.4. We then have

$$\begin{aligned} u_4(A_3u_3A_3 + A_3s_3s_2^{-1}s_3A_3)u_4u_3u_2u_3u_4 & \subset A_3u_4u_3A_3u_4u_3u_2u_3u_4 + A_3u_4s_3s_2^{-1}s_3A_3u_4u_3u_2u_3u_4 \\ & \subset A_3u_4u_3(u_1u_2u_1u_2)u_4u_3u_2u_3u_4 + A_3u_4s_3s_2^{-1}s_3(u_1u_2u_1u_2)u_4u_3u_2u_3u_4 \\ & \subset A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 + A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4(u_2u_3u_2u_3)u_4 \\ & \subset A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 + A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4u_3u_2u_3u_2u_4 \\ & \subset A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 + A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4u_3u_2u_3u_4u_2 \\ & \subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 + A_4u_4A_4u_4A_4 \end{aligned}$$

by Lemmas 6.3 and 6.4, and $u_4A_3(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_4u_3u_2u_3u_4 = A_3u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4$ by Proposition 6.3, so this solves the case $\alpha = \beta$.

We can thus assume $\alpha = -\beta$, that is we consider the expression $u_4s_3^{\beta}u_2u_1u_2s_3^{-\beta}u_2u_4u_1u_3u_2u_3u_4$, that we split in two cases $u_4s_3^{\beta}u_2u_1u_2s_3^{-\beta}s_2^{\gamma}u_4u_1u_3u_2u_3u_4$ for $\gamma \in \{-1, 1\}$. Up to applying Φ, we can restrict to $u_4s_3^{\beta}u_2u_1u_2s_3^{-\beta}s_2^{\gamma}s_4u_1u_3u_2u_3s_4^{\alpha}$ for some $\alpha \in \{-1, 1\}$, and using $u_3u_2u_3 \subset s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}u_2 + u_2u_3u_2$ we can restrict to $u_4s_3^{\beta}u_2u_1u_2s_3^{-\beta}s_2^{\gamma}s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha}$ by Proposition 6.3.

First assume $\gamma = -1$. Using again $u_2u_1u_2 \subset u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ we can restrict to studying $u_4s_3^{\beta}s_2s_1^{-1}s_2s_3^{-\beta}s_2^{-1}s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha}$. If $\beta = 1$, then we get

$$\begin{aligned} u_4s_3s_2s_1^{-1}s_2s_3^{-1}s_2^{-1}s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} & \subset u_4s_2^{-1}s_3s_2s_1^{-1}s_2^{-1}s_3s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset s_2^{-1}u_4s_3s_2s_1^{-1}s_2^{-1}s_3s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset A_4u_4A_4u_4A_4 \end{aligned}$$

by Proposition 6.3. For the case $\beta = -1$, we can restrict to an expression of the form $u_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_2^{-1} s_4 u_1 s_3^{\alpha} s_2^{-\alpha} s_3^{\alpha} s_4^{\alpha}$, and we get

$$\begin{aligned} u_4 s_3^{-1} s_2 s_1^{-1} (s_2 s_3 s_2^{-1}) s_4 u_1 s_3^{\alpha} s_2^{-\alpha} s_3^{\alpha} s_4^{\alpha} &\subset u_4 s_3^{-1} s_2 s_1^{-1} s_3^{-1} s_2 s_3 s_4 u_1 s_3^{\alpha} s_2^{-\alpha} s_3^{\alpha} s_4^{\alpha} \\ &\subset u_4 s_3^{-1} s_2 s_1^{-1} s_3^{-1} s_2 s_3 s_4 u_1 s_3^{\alpha} s_2^{-\alpha} s_3^{\alpha} s_4^{\alpha} \\ &\subset u_4 s_3^{-1} s_2 s_1^{-1} s_3^{-1} s_2 (s_3 s_4 s_3^{\alpha}) u_1 s_2^{-\alpha} s_3^{\alpha} s_4^{\alpha} \\ &\subset u_4 s_3^{-1} s_2 s_1^{-1} s_3^{-1} s_2 s_4^{\alpha} s_3 s_4 u_1 s_2^{-\alpha} s_3^{\alpha} s_4^{\alpha} \\ &\subset u_4 s_3^{-1} s_2 s_1^{-1} s_3^{-1} s_2 s_4^{\alpha} s_3 u_1 s_2^{-\alpha} s_4 s_3^{\alpha} s_4^{\alpha}) \\ &\subset u_4 s_3^{-1} s_2 s_1^{-1} s_3^{-1} s_2 s_4^{\alpha} s_3 u_1 s_2^{-\alpha} s_3^{\alpha} s_4^{\alpha} s_3 \\ &\subset A_4 (u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4) A_4 + A_4 u_4 A_4 u_4 A_4 \end{aligned}$$

by Proposition 6.3 and Lemma 6.4.

Now assume $\gamma = 1$. Using again $u_2 u_1 u_2 \subset u_1 s_2^{-1} s_1 s_2^{-1} + u_1 u_2 u_1$ we can restrict to the form $u_4 s_3^{\beta} s_2^{-1} s_1 s_2^{-1} s_3^{-\beta} s_2 u_4 u_1 u_3 u_2 u_3 u_4$. If $\beta = 1$ we get

$$\begin{aligned} u_4 s_3 s_2^{-1} s_1 (s_2^{-1} s_3^{-1} s_2) u_4 u_1 u_3 u_2 u_3 u_4 &\subset u_4 s_3 s_2^{-1} s_1 s_3 s_2^{-1} s_3^{-1} u_4 u_1 u_3 u_2 u_3 u_4 \\ &\subset u_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_3^{-1} u_4 u_1 u_3 u_2 u_3 u_4 \\ &\subset u_4 u_2 (s_3^{-1} s_2 s_3^{-1}) s_1 s_2^{-1} s_3^{-1} u_4 u_1 u_3 u_2 u_3 u_4 + u_4 u_2 u_3 u_2 s_1 s_2^{-1} s_3^{-1} u_4 u_1 u_3 u_2 u_3 u_4 \\ &\subset u_2 u_4 s_3^{-1} s_2 s_3^{-1} s_1 s_2^{-1} s_3^{-1} u_4 u_1 u_3 u_2 u_3 u_4 + u_2 u_4 u_3 u_2 s_1 s_2^{-1} s_3^{-1} u_4 u_1 u_3 u_2 u_3 u_4 \\ &\subset u_2 u_4 s_3^{-1} s_2 s_1 (s_3^{-1} s_2^{-1} s_3^{-1}) u_4 u_1 u_3 u_2 u_3 u_4 + A_4 u_4 A_4 u_4 A_4 \\ &\subset u_2 u_4 s_3^{-1} (s_2 s_1 s_2^{-1}) s_3^{-1} s_2^{-1} u_4 u_1 u_3 u_2 u_3 u_4 + A_4 u_4 A_4 u_4 A_4 \\ &\subset u_2 u_4 s_3^{-1} s_1^{-1} s_2 s_1 s_3^{-1} s_2^{-1} u_4 u_1 u_3 u_2 u_3 u_4 + A_4 u_4 A_4 u_4 A_4 \\ &\subset u_2 s_1^{-1} u_4 s_3^{-1} s_2 s_1 s_3^{-1} s_2^{-1} u_4 u_1 u_3 u_2 u_3 u_4 + A_4 u_4 A_4 u_4 A_4 \\ &\subset A_4 (u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4) A_4 + A_4 u_4 A_4 u_4 A_4 \end{aligned}$$

by Proposition 6.3 and Lemma 6.4.

If $\beta = -1$ we get that $u_4 s_3^{-1} s_2^{-1} s_1 s_2^{-1} s_3 s_2 u_4 u_1 u_3 u_2 u_3 u_4$ is included into

$$u_4 s_2 s_3^{-1} s_2^{-1} s_1 s_2 s_3^{-1} u_4 u_1 u_3 u_2 u_3 u_4 \subset s_2 u_4 s_3^{-1} s_2^{-1} s_1 s_2 s_3^{-1} u_4 u_1 u_3 u_2 u_3 u_4 \subset A_4 u_4 A_4 u_4 A_4$$

by Proposition 6.3. This concludes the proof of the lemma. $\square$

Proposition 6.6. *$u_4 A_4 u_4 A_4 u_4 \subset A_4 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_4 + A_4 u_4 A_4 u_4 A_4$, and thus $A_5^{(3)} \subset A_4 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_4 + A_4 u_4 A_4 u_4 A_4$.*

Proof. From Lemmas 6.4 and 6.5 it is sufficient to prove

$$\begin{aligned} u_4 A_4 u_4 A_4 u_4 \subset\ & A_4 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 + A_4 u_4 A_4 u_4 A_4 \\ & + A_3 u_4 u_3 u_2 u_3 u_1 u_2 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_2 u_1 u_3 u_2 u_3 u_4 A_3 \end{aligned}$$

By Theorem 4.1 we have $A_4 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 A_3 + A_3 u_3 u_2 u_1 u_2 u_3$ and $A_4 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 A_3 + u_3 u_2 u_1 u_2 u_3 A_3$, whence

$$\begin{aligned} u_4 A_4 u_4 A_4 u_4 \subset\ & u_4 A_3 u_3 A_3 u_4 A_3 u_3 A_3 u_4 + u_4 A_3 u_3 A_3 u_4 A_3 u_3 u_2 u_3 A_3 u_4 \\ & + u_4 A_3 u_3 A_3 u_4 u_3 u_2 u_1 u_2 u_3 A_3 u_4 + u_4 A_3 u_3 u_2 u_3 A_3 u_4 A_3 u_3 A_3 u_4 \\ & + u_4 A_3 u_3 u_2 u_3 A_3 u_4 A_3 u_3 u_2 u_3 A_3 u_4 + u_4 A_3 u_3 u_2 u_3 A_3 u_4 u_3 u_2 u_1 u_2 u_3 A_3 u_4 \\ & + u_4 A_3 u_3 u_2 u_1 u_2 u_3 u_4 A_3 u_3 A_3 u_4 + u_4 A_3 u_3 u_2 u_1 u_2 u_3 u_4 A_3 u_3 u_2 u_3 A_3 u_4 \\ & + u_4 A_3 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 A_3 u_4 \\ \subset\ & A_3 u_4 u_3 A_3 u_4 u_3 u_4 A_3 + A_3 u_4 u_3 A_3 u_4 u_3 u_2 u_3 u_4 A_3 \\ & + A_3 u_4 u_3 A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 + A_3 u_4 u_3 u_2 u_3 u_4 A_3 u_3 u_4 A_3 \\ & + A_3 u_4 u_3 u_2 u_3 A_3 u_4 u_3 u_2 u_3 u_4 A_3 + A_3 u_4 u_3 u_2 u_3 A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\ & + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 u_3 u_4 A_3 + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 u_3 u_2 u_3 u_4 A_3 \\ & + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\ \subset\ & A_3 u_4 u_3 A_3 u_4 u_3 u_4 A_3 + A_3 u_4 u_3 A_3 u_4 u_3 u_2 u_3 u_4 A_3 \\ & + A_3 u_4 u_3 A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 + A_3 u_4 u_3 u_2 u_3 u_4 A_3 u_3 u_4 A_3 \\ & + A_3 u_4 u_3 u_2 u_3 A_3 u_4 u_3 u_2 u_3 u_4 A_3 + A_3 u_4 u_3 u_2 u_3 A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\ & + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 u_3 u_4 A_3 + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 u_3 u_2 u_3 u_4 A_3 \\ & + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3. \end{aligned}$$

We have

(1) $A_3 u_4 u_3 A_3 u_4 u_3 u_4 A_3 \subset A_3 u_4 u_3 (u_2 u_1 u_2 u_1) u_4 u_3 u_4 A_3 \subset A_4 u_4 A_4 u_4 A_4$ by Proposition 6.3.
(2) $A_3 u_4 u_3 A_3 u_4 u_3 u_2 u_3 u_4 A_3 \subset A_3 u_4 u_3 u_2 u_1 u_2 u_1 u_4 u_3 u_2 u_3 u_4 A_3 \subset A_4 u_4 A_4 u_4 A_4$ by Proposition 6.3.

(3) We have

$$\begin{aligned} A_3u_4u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 &\subset A_3u_4u_3u_2u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_1u_2u_4u_3(u_1u_2u_1u_2)u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_1u_2u_4u_3u_2u_1u_2u_1u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_1u_2u_4u_3u_2u_1u_2u_3u_4A_3 \\ &\subset A_4u_4A_4u_4A_4 \end{aligned}$$

by Proposition 6.3.

(4) $A_3u_4u_3u_2u_3u_4A_3u_3u_4A_3 \subset A_3u_4u_3u_2u_3u_4u_2u_1u_2u_1u_3u_4A_3 \subset A_4u_4A_4u_4A_4$ by Proposition 6.3.

(5) Using $A_3 = u_2u_1u_2u_1$ we get

$$\begin{aligned} A_3u_4u_3u_2u_3A_3u_4u_3u_2u_3u_4A_3 &\subset A_3u_4(u_3u_2u_3u_2)u_1u_2u_1u_4u_3u_2u_3u_4A_3 \\ &\subset A_3u_4u_2u_3u_2u_3u_1u_2u_1u_4u_3u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_3u_1u_2u_1u_4u_3u_2u_3u_4A_3 \\ &\subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_3 + A_4u_4A_4u_4A_4 \end{aligned}$$

by Lemma 6.4.

(6) Using $A_3 = u_2u_1u_2u_1$ we get

$$\begin{aligned} A_3u_4u_3u_2u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 &\subset A_3u_4u_3u_2u_3u_2u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ &\subset A_3u_4(u_3u_2u_3u_2)u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ &\subset A_3u_4u_2u_3u_2u_3u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_3u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_3u_1u_2u_4u_3u_1u_2u_1u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_3u_1u_2u_4u_3(u_1u_2u_1u_2)u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_3u_1u_2u_4u_3u_2u_1u_2u_1u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_3u_1u_2u_4u_3u_2u_1u_2u_3u_4A_3. \end{aligned}$$

(7) Using $A_3 = u_1u_2u_1u_2$ we get similarly

$$A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_4A_3 \subset A_3u_4u_3u_2u_1u_2u_3u_4u_2u_1u_2u_3u_4A_3 \subset A_4u_4A_4u_4A_4$$

by Proposition 6.3, and

(8)

$$\begin{aligned} A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_2u_3u_4A_3 &\subset A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1u_2u_3u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1(u_2u_3u_2u_3)u_4A_3 \\ &\subset A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1u_3u_2u_3u_2u_4A_3 \\ &\subset A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1u_3u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_1u_2u_1u_3u_4u_2u_1u_3u_2u_3u_4A_3 \\ &\subset A_3u_4u_3(u_2u_1u_2u_1)u_3u_4u_2u_1u_3u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_1u_2u_1u_2u_3u_4u_2u_1u_3u_2u_3u_4A_3 \\ &\subset A_3u_4u_3u_2u_1u_2u_3u_4u_2u_1u_3u_2u_3u_4A_3 \end{aligned}$$

(9) The case $A_3u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_3$ is clear. □

6.2. The A_4-bimodule $A_5^{(3)}/A_5^{(2)}$: a smaller set of generators.

Lemma 6.7. *For all $\alpha, \beta, \gamma, \ldots \in \{-1, 1\}$,*

$$\begin{aligned} s_4^\alpha s_3^\beta A_3 s_3^\gamma s_4^\delta s_3^\varepsilon A_3 s_3^\zeta s_4^\eta &\subset u_1 s_4^\alpha s_3^\beta (s_2 s_1^{-1} s_2) s_3^\gamma s_4^\delta s_3^\varepsilon (s_2 s_1^{-1} s_2) s_3^\zeta s_4^\eta u_1 + A_5^{(2)} \\ s_4^\alpha s_3^\beta A_3 s_3^\gamma s_4^\delta s_3^\varepsilon A_3 s_3^\zeta s_4^\eta &\subset u_1 s_4^\alpha s_3^\beta (s_2^{-1} s_1 s_2^{-1}) s_3^\gamma s_4^\delta s_3^\varepsilon (s_2 s_1^{-1} s_2) s_3^\zeta s_4^\eta u_1 + A_5^{(2)} \\ s_4^\alpha s_3^\beta A_3 s_3^\gamma s_4^\delta s_3^\varepsilon A_3 s_3^\zeta s_4^\eta &\subset u_1 s_4^\alpha s_3^\beta (s_2 s_1^{-1} s_2) s_3^\gamma s_4^\delta s_3^\varepsilon (s_2^{-1} s_1 s_2^{-1}) s_3^\zeta s_4^\eta u_1 + A_5^{(2)} \\ s_4^\alpha s_3^\beta A_3 s_3^\gamma s_4^\delta s_3^\varepsilon A_3 s_3^\zeta s_4^\eta &\subset u_1 s_4^\alpha s_3^\beta (s_2^{-1} s_1 s_2^{-1}) s_3^\gamma s_4^\delta s_3^\varepsilon (s_2^{-1} s_1 s_2^{-1}) s_3^\zeta s_4^\eta u_1 + A_5^{(2)} \end{aligned}$$

Proof. This is an easy consequence of the decompositions $A_3 = u_1u_2u_1 + u_1s_2s_1^{-1}s_2 = u_1u_2u_1 + s_2s_1^{-1}s_2u_1 = u_1u_2u_1 + u_1s_2^{-1}s_1s_2^{-1} = u_1u_2u_1 + s_2^{-1}s_1s_2^{-1}u_1$ of Theorem 3.2 and of Proposition 6.3. □

Lemma 6.8. *For $i, j, k, \alpha, \beta, \gamma \in \{-1, 1\}$,*

(1) $s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\beta A_3 s_3^\gamma s_4^k \subset A_5^{(2)}$ *unless $i = j = k$*
(2) $s_4^i s_3^\alpha A_3 s_3^\beta s_4^j s_3^\gamma A_3 s_3^{-\gamma} s_4^k \subset A_5^{(2)}$ *unless $i = j = k$*
(3) $s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\beta A_3 s_3^\beta s_4^k \subset A_5^{(2)}$
(4) $s_4^i s_3^\alpha A_3 s_3^\alpha s_4^j s_3^\beta A_3 s_3^{-\beta} s_4^k \subset A_5^{(2)}$
(5) $s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\alpha A_3 s_3^{-\alpha} s_4^k \subset A_5^{(2)}$.

Proof. We use the formulas $s_3^{-1}(s_2s_1^{-1}s_2)s_3 = s_2s_1(s_3s_2^{-1}s_3)s_1^{-1}s_2^{-1}$ and $s_3(s_2s_1^{-1}s_2)s_3^{-1} = s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_1s_2$ which are easy to prove and which already hold in the braid group B_4, and can be summarized as $s_3^{-\alpha}(s_2s_1^{-1}s_2)s_3^{\alpha} = s_2^{\alpha}s_1^{\alpha}(s_3s_2^{-1}s_3)s_1^{-\alpha}s_2^{-\alpha}$ for $\alpha \in \{-1, 1\}$. We also use the fact that s_2 (and thus s_2^{-1}) commutes with $s_3s_2s_1^{-1}s_2s_3$ (already in the braid group B_4), and similarly s_2^{-1} (and thus s_2) commutes with $s_3^{-1}s_2^{-1}s_1s_2^{-1}s_3^{-1}$. Together with Lemma 6.7, this yields

$$\begin{aligned}
s_4^is_3^{\alpha}A_3s_3^{-\alpha}s_4^js_3^{\beta}A_3s_3^{\beta}s_4^k &\subset s_4^is_3^{\alpha}(s_2s_1^{-1}s_2)s_3^{-\alpha}s_4^js_3^{\beta}(s_2^{\beta}s_1^{-\beta}s_2^{\beta})s_3^{\beta}s_4^k + A_5^{(2)} \\
&\subset s_4^i(s_3^{\alpha}s_2s_1^{-1}s_2s_3^{-\alpha})s_4^j(s_3^{\beta}s_2^{\beta}s_1^{-\beta}s_2^{\beta}s_3^{\beta})s_4^k + A_5^{(2)} \\
&\subset s_4^is_2^{-\alpha}s_1^{-\alpha}(s_3s_2^{-1}s_3)s_1^{\alpha}s_2^{\alpha}s_4^j(s_3^{\beta}s_2^{\beta}s_1^{-\beta}s_2^{\beta}s_3^{\beta})s_4^k + A_5^{(2)} \\
&\subset s_2^{-\alpha}s_1^{-\alpha}s_4^is_3s_2^{-1}s_3s_1^{\alpha}s_4^js_2^{\alpha}(s_3^{\beta}s_2^{\beta}s_1^{-\beta}s_2^{\beta}s_3^{\beta})s_4^k + A_5^{(2)} \\
&\subset A_3s_4^is_3s_2^{-1}s_3s_1^{\alpha}s_4^j(s_3^{\beta}s_2^{\beta}s_1^{-\beta}s_2^{\beta}s_3^{\beta})s_2^{\alpha}s_4^k + A_5^{(2)} \\
&\subset A_3s_4^is_3s_2^{-1}s_3s_1^{\alpha}s_4^j(s_3^{\beta}s_2^{\beta}s_1^{-\beta}s_2^{\beta}s_3^{\beta})s_4^kA_3 + A_5^{(2)} \\
&\subset A_5^{(2)}
\end{aligned}$$

by Proposition 6.3, and this proves (3), as well as the symmetric case (4). This also proves (1) in case $\beta = \gamma$. We thus deal with

$$\begin{aligned}
s_4^is_3^{\alpha}A_3s_3^{-\alpha}s_4^js_3^{\beta}A_3s_3^{-\beta}s_4^k &\subset s_4^is_3^{\alpha}(s_2s_1^{-1}s_2)s_3^{-\alpha}s_4^js_3^{\beta}(s_2s_1^{-1}s_2)s_3^{-\beta}s_4^k + A_5^{(2)} \\
&\subset s_4^i(s_3^{\alpha}s_2s_1^{-1}s_2s_3^{-\alpha})s_4^j(s_3^{\beta}s_2s_1^{-1}s_2s_3^{-\beta})s_4^k + A_5^{(2)} \\
&\subset s_4^is_2^{-\alpha}s_1^{-\alpha}(s_3s_2^{-1}s_3)s_1^{\alpha}s_2^{\alpha}s_4^js_2^{-\beta}s_1^{-\beta}(s_3s_2^{-1}s_3)s_1^{\beta}s_2^{\beta}s_4^k + A_5^{(2)} \\
&\subset s_2^{-\alpha}s_1^{-\alpha}s_4^is_3s_2^{-1}s_3s_1^{\alpha}s_2^{\alpha}s_2^{-\beta}s_4^js_1^{-\beta}(s_3s_2^{-1}s_3)s_4^ks_1^{\beta}s_2^{\beta} + A_5^{(2)} \\
&\subset A_5^{(2)}
\end{aligned}$$

if $\alpha = \beta$ by Proposition 6.3, and we get (5). Otherwise, $\alpha = -\beta$, and

$$\begin{aligned}
s_4^is_3^{\alpha}A_3s_3^{-\alpha}s_4^js_3^{\beta}A_3s_3^{-\beta}s_4^k &\subset s_2^{-\alpha}s_1^{-\alpha}s_4^is_3s_2^{-1}s_3s_1^{\alpha}s_2^{2\alpha}s_1^{\alpha}s_4^j(s_3s_2^{-1}s_3)s_4^ks_1^{-\alpha}s_2^{-\alpha} + A_5^{(2)} \\
&\subset A_5^{(2)}
\end{aligned}$$

unless $i = j = k$ by Proposition 6.3, and we get (1). (2) is proved symmetrically. $\square$

Corollary 6.9. (1) $s_4s_3^{-1}(s_2s_1^{-1}s_2)s_3s_4^{-1}s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4 \in A_4u_4A_4u_4A_4$
(2) $s_4^{-1}s_3(s_2s_1^{-1}s_2)s_3s_4^{-1}s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4 \in A_4u_4A_4u_4A_4$.

Lemma 6.10. $s_4^{-1}w^+s_4^{-1}w^+s_4^{-1} \in A_4s_4w^-s_4w^-s_4A_4 + A_4u_4A_4u_4A_4$.

Proof. We first use $s_2^{-1}s_1s_2^{-1} \in u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ and $s_2^{-1}s_1s_2^{-1} \in s_2s_1^{-1}s_2u_1 + u_1u_2u_1$ together with Proposition 6.3 to get

$$\begin{aligned}
s_4^{-1}w^+s_4^{-1}w^+s_4^{-1} &= s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} \\
&\subset u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} + u_1s_4^{-1}s_3u_2u_1s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} \\
&\subset u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1} + A_4u_4A_4u_4A_4 \\
&\subset u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}u_1 + A_4u_4A_4u_4A_4 \\
&\subset A_4(s_3^{-1}s_4^{-1}s_3)s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2(s_3s_4^{-1}s_3^{-1})A_4 + A_4u_4A_4u_4A_4 \\
&\subset A_4s_4s_3^{-1}s_4^{-1}s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_4^{-1}s_3^{-1}s_4A_4 + A_4u_4A_4u_4A_4 \\
&\subset A_4s_4s_3^{-1}s_2s_1^{-1}s_2(s_4^{-1}s_3s_4^{-1}s_3s_4^{-1})s_2s_1^{-1}s_2s_3^{-1}s_4A_4 + A_4u_4A_4u_4A_4
\end{aligned}$$

By Lemma 3.7 $s_4^{-1}s_3s_4^{-1}s_3s_4^{-1}$ is a linear combination of terms of several kinds

(1) elements x of u_3u_4 or u_4u_3, for which we get $s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ by a direct application of Proposition 6.3.
(2) elements x that can be put in the form $s_4^{\alpha}s_3^{\beta}s_4^{\gamma}$ with $\alpha = -1$ or $\gamma = -1$, in which case we get $s_4s_3^{-1}s_2s_1^{-1}$ $s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ through one application of the equation $s_4s_3^{\cdots}s_4^{-1} \in u_3u_4u_3$ or $s_4^{-1}s_3^{\cdots}s_4 \in u_3u_4u_3$, and Proposition 6.3.
(3) the element $s_3^{-1}s_4s_3^{-1}$, which provides $s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4 = s_4w^-s_4w^-s_4w^-$.
(4) the element $x = s_3s_4^{-1}s_3$, for which we get $s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ by Corollary 6.9 (1).
(5) the element $x = s_4^{-1}s_3s_4^{-1}s_3$, for which we get

$$\begin{aligned}
s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 &= s_4s_3^{-1}s_2s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&= (s_4s_3^{-1}s_4^{-1})s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&= s_3^{-1}s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&\subset A_4s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&\subset A_4u_4A_4u_4A_4
\end{aligned}$$

by Corollary 6.9(2).

This proves the inclusion. $\square$

Lemma 6.11. (1) $u_4A_4u_4u_3u_4 \subset A_4u_4A_4u_4A_4$
(2) $u_4u_3u_4A_4u_4 \subset A_4u_4A_4u_4A_4$
(3) $s_4^{\beta}u_3u_2u_1u_2s_3^{\alpha}s_4^{\gamma}s_3^{-\alpha}u_2u_1u_2u_3s_4^{\beta} \subset A_4u_4A_4u_4A_4$
(4) $s_4^{\alpha}s_3^{\alpha}u_2u_1u_2s_3^{\alpha}s_4^{\gamma}s_3^{-\alpha}u_2u_1u_2u_3s_4^{\beta} \subset A_4u_4A_4u_4A_4$
(5) $s_4^{\beta}u_3u_2u_1u_2s_3^{\alpha}s_4^{\gamma}s_3^{-\alpha}u_2u_1u_2s_3^{-\alpha}s_4^{-\alpha} \subset A_4u_4A_4u_4A_4$
(6) $u_4s_3^{\alpha}u_2u_1u_2s_3^{\alpha}s_4^{\alpha}s_3^{\alpha}u_2u_1u_2s_3^{\alpha}u_4 \subset A_4u_4A_4u_4A_4$
(7) $s_4w^+s_4^{-1}w^+s_4^{-1} \in A_4u_4A_4u_4A_4$
(8) $s_4w^-s_4w^-s_4^{-1} \in A_4u_4A_4u_4A_4$.

Proof. Since $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ we have $u_4A_4u_4u_3u_4 \subset A_3u_4u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4$. We have $u_4u_3A_3u_4u_3u_4 \subset u_4u_3u_1u_2u_1u_2u_4u_3u_4 \subset A_4u_4A_4u_4A_4$ by Proposition 6.3,

$$\begin{aligned} & u_4u_3u_2u_3A_3u_4u_3u_4 && \subset u_4(u_3u_2u_3u_2)u_1u_2u_1u_4u_3u_4 \\ = \; & u_4u_2u_3u_2u_3u_1u_2u_1u_4u_3u_4 && = u_2u_4u_3u_2u_3u_1u_4u_2u_1u_3u_4 \subset A_4u_4A_4u_4A_4 \end{aligned}$$

by Proposition 6.3, and $u_4u_3u_2u_1u_2u_3u_4u_3u_4 \subset A_4u_4A_4u_4A_4$ by Proposition 6.3. This proves (1). (2) is deduced from (1) by applying Ψ. We turn to (3). Since $s_4^{\beta}u_3u_2u_1u_2(s_3^{\alpha}s_4^{\gamma}s_3^{-\alpha})u_2u_1u_2u_3s_4^{\beta} = s_4^{\beta}u_3u_2u_1u_2s_4^{-\alpha}s_3^{\gamma}s_4^{\alpha}u_2u_1u_2u_3s_4^{\beta} = s_4^{\beta}u_3s_4^{-\alpha}u_2u_1u_2 s_3^{\gamma}u_2u_1u_2s_4^{\alpha}u_3s_4^{\beta}$ and either $s_4^{\beta}u_3s_4^{-\alpha} \subset u_3u_4u_3$ or $s_4^{\alpha}u_3s_4^{\beta} \subset u_3u_4u_3$. In both cases we get an element of $A_4u_4A_4u_4u_3u_4A_4 \subset A_4u_4A_4u_4A_4$ or $A_4u_4u_3u_4A_4u_4A_4 \subset A_4u_4A_4u_4A_4$ by (1) or (2), and this proves (3). (4) and (5) are similar and left to the reader.

Now

$$\begin{aligned} u_4s_3^{\alpha}u_2u_1u_2s_3^{\alpha}s_4^{\alpha}s_3^{\alpha}u_2u_1u_2s_3^{\alpha}u_4 &= u_4s_3^{\alpha}u_2u_1u_2s_4^{\alpha}s_3^{\alpha}s_4^{\alpha}u_2u_1u_2s_3^{\alpha}u_4 \\ &= (u_4s_3^{\alpha}s_4^{\alpha})u_2u_1u_2s_3^{\alpha}u_2u_1u_2(s_4^{\alpha}s_3^{\alpha}u_4) \\ &\subset u_3u_4u_3u_2u_1u_2u_3u_2u_1u_2u_3u_4u_3 \quad \subset A_4u_4A_4u_4A_4 \end{aligned}$$

and this proves (6). In order to prove (7), we compute, using b, b' for elements in $u_2u_1u_2$,

$$\begin{aligned} s_4w^+s_4^{-1}w^+s_4^{-1} &\subset s_4s_3bs_3s_4^{-1}s_3b's_3s_4^{-1} \\ &\subset s_3^{-1}(s_3s_4s_3)bs_3s_4^{-1}s_3b's_3s_4^{-1} \\ &\subset s_3^{-1}s_4s_3s_4bs_3s_4^{-1}s_3b's_3s_4^{-1} \\ &\subset s_3^{-1}s_4s_3b(s_4s_3s_4^{-1})s_3b's_3s_4^{-1} \\ &\subset s_3^{-1}s_4s_3bs_3^{-1}s_4s_3^2b's_3s_4^{-1} \end{aligned}$$

Now $s_3^2 \in R + Rs_3 + Rs_3^{-1}$, and $s_3^{-1}s_4s_3bs_3^{-1}s_4b's_3s_4^{-1} \in A_4u_4A_4u_4A_4$ by Proposition 6.3,

$$s_3^{-1}s_4s_3bs_3^{-1}s_4s_3b's_3s_4^{-1} \in A_4u_4A_4u_4A_4$$

by Lemma 6.8(3), and $s_3^{-1}s_4s_3bs_3^{-1}s_4s_3^{-1}b's_3s_4^{-1} \subset A_5^{(2)}$ by Lemma 6.8(1).

The proof of (8) is similar and left to the reader. □

Lemma 6.12. (1) $s_4s_3^{-1}A_3s_3s_4s_3A_3s_3^{-1}s_4 \subset u_3s_4^{-}w^+s_4w^-s_4 + A_5^{(2)}$
(2) $(s_4w^+s_4^{-1}w^+s_4)s_3^{-1} \in s_3^{-1}(s_4w^+s_4^{-1}w^+s_4) + A_5^{(2)}$
(3) $s_4w^+s_4^{-1}w^+s_4 \in A_3^{\times}s_4(s_3s_2^{-1}s_3)(s_1s_2^{-1}s_1)s_4(s_3s_2^{-1}s_3)s_4A_4^{\times} + A_5^{(2)}$
(4) $s_4w^-s_4w^+s_4^{-1} \in A_4^{\times}s_4^{-1}w^+s_4w^-s_4A_4^{\times} + A_5^{(2)}$
(5) $s_4w^-s_4^{-1}w^+s_4^{-1} \in A_4^{\times}s_4^{-1}w^+s_4^{-1}w^-s_4A_4^{\times} + A_5^{(2)}$
(6) $s_4s_3A_3s_3^{-1}s_4s_3^{-1}A_3s_3s_4 \subset u_3s_4w^+s_4^{-1}w^+s_4u_3 + A_5^{(2)}$.

Proof. We first prove (1). By Lemma 6.7 we need to prove $s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \subset u_3s_4^{-}w^+s_4w^-s_4 + A_5^{(2)}$, and we get, using Proposition 6.3

$$\begin{aligned} s_4s_3^{-1}s_2s_1^{-1}s_2(s_3s_4s_3)s_2s_1^{-1}s_2s_3^{-1}s_4 &= s_4s_3^{-1}s_2s_1^{-1}s_2s_4s_3s_4s_2s_1^{-1}s_2s_3^{-1}s_4 \\ &= (s_4s_3^{-1}s_4)s_2s_1^{-1}s_2s_3s_4s_2s_1^{-1}s_2s_3^{-1}s_4 \\ &\subset u_3s_4^{-1}s_3s_4^{-1}s_2s_1^{-1}s_2s_3s_4s_2s_1^{-1}s_2s_3^{-1}s_4 + u_3u_4u_3s_2s_1^{-1}s_2s_3s_4s_2s_1^{-1}s_2s_3^{-1}s_4 \\ &\subset u_3s_4^{-1}s_3s_4^{-1}s_2s_1^{-1}s_2s_3s_4s_2s_1^{-1}s_2s_3^{-1}s_4 + A_5^{(2)} \\ &\subset u_3s_4^{-1}s_3s_2s_1^{-1}s_2(s_4^{-1}s_3s_4)s_2s_1^{-1}s_2s_3^{-1}s_4 + A_5^{(2)} \\ &\subset u_3s_4^{-1}s_3(s_2s_1^{-1}s_2)s_3s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4 + A_5^{(2)} \\ &\subset u_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4 + A_5^{(2)} \\ &\subset u_3s_4^{-1}w^+s_4w^-s_4 + A_5^{(2)}. \end{aligned}$$

We now prove (2). We have, using $s_3s_4^{-1}s_3s_4^{-1} \in Rs_4s_3^{-1}s_4s_3^{-1} + u_3u_4u_3 + u_4u_3u_4$ and $s_3s_4^{-1}s_3s_4^{-1} - s_4^{-1}s_3s_4^{-1}s_3 \in u_3u_4 + u_4u_3$ by Lemma 3.5, we get

$$\begin{aligned}
s_4w^+s_4^{-1}w^+s_4.s_3^{-1} &= s_4w^+s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}(s_3s_4s_3^{-1}) \\
&= s_4w^+s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_4^{-1}s_3s_4 \\
&= s_4w^+s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1s_2^{-1}s_3s_4 \\
&= s_4s_3s_2^{-1}s_1s_2^{-1}(s_3s_4^{-1}s_3s_4^{-1})s_2^{-1}s_1s_2^{-1}s_3s_4 \\
&\in s_4s_3s_2^{-1}s_1s_2^{-1}s_4^{-1}s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4 + A_5^{(2)} \\
&\subset (s_4s_3s_4^{-1})s_2^{-1}s_1s_2^{-1}s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4 + A_5^{(2)} \\
&\subset s_3^{-1}s_4s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4 + A_5^{(2)} \\
&\subset s_3^{-1}.s_4w^+s_4^{-1}w^+s_4 + A_5^{(2)}
\end{aligned}$$

We now prove (3). We have

$$\begin{aligned}
s_4w^+s_4^{-1}w^+s_4 &= s_4s_3(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1}s_3(s_2^{-1}s_1s_2^{-1})s_3s_4 \\
&\in s_4s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4 + A_5^{(2)} \\
&\subset s_3^{-1}s_4s_3s_2s_1^{-1}s_2s_4s_3^2s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset Rs_3^{-1}s_4s_3s_2s_1^{-1}s_2s_4^2s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + Rs_3^{-1}s_4s_3s_2s_1^{-1}s_2(s_4s_3s_4)s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} \\
&\quad + R^\times s_3^{-1}s_4s_3s_2s_1^{-1}s_2s_4s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset Rs_3^{-1}s_4s_3s_2s_1^{-1}s_2(s_3s_4s_3)s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + R^\times s_3^{-1}s_4s_3s_2s_1^{-1}s_2s_4s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset Rs_3^{-1}s_4s_3s_2s_1^{-1}s_2s_3s_4s_2s_1^{-1}s_2s_3s_4s_3^{-1} + R^\times s_3^{-1}s_4s_3s_2s_1^{-1}s_2s_4s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_3^{-1}s_4s_3s_2s_1^{-1}s_2s_4s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_3^{-1}(s_4s_3s_4)s_2s_1^{-1}s_2s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_3^{-1}s_3s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_4(s_3s_2s_1^{-1}s_2s_3^{-1})s_4(s_3^{-1}s_2s_1^{-1}s_2s_3)s_4s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_4s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_1s_2s_4s_2s_1(s_3s_2^{-1}s_3)s_1^{-1}s_2^{-1}s_4s_3^{-1} + A_5^{(2)} \\
&\subset A_3^\times s_4(s_3s_2^{-1}s_3)s_1s_2s_2s_1s_4(s_3s_2^{-1}s_3)s_4s_1^{-1}s_2^{-1}s_3^{-1} + A_5^{(2)}
\end{aligned}$$

and then $s_1s_2s_2s_1 = s_1s_2^2s_1 \in R^\times s_1s_2^{-1}s_1 + Rs_1s_2s_1 + Rs_1^2$. Since $s_4(s_3s_2^{-1}s_3)s_1s_2s_1s_4(s_3s_2^{-1}s_3)s_4 = s_4(s_3s_2^{-1}s_3)s_2s_1s_2s_4(s_3s_2^{-1}s_3)s_4 \subset s_4(u_3u_2u_3u_2)s_1s_2s_4u_3u_2u_3s_4 = s_4u_2u_3u_2u_3s_1s_2s_4u_3u_2u_3s_4 = u_2s_4u_3u_2u_3s_1s_2s_4u_3u_2u_3s_4 \subset A_5^{(2)}$ by Proposition 6.3 and similarly $s_4(s_3s_2^{-1}s_3)s_1s_1s_4(s_3s_2^{-1}s_3)s_4 \in s_4(s_3s_2^{-1}s_3)u_1s_4(s_3s_2^{-1}s_3)s_4 \subset A_5^{(2)}$, this proves that the element $s_4w^+s_4^{-1}w^+s_4$ belongs to the set $A_3^\times s_4(s_3s_2^{-1}s_3)(s_1s_2^{-1}s_1)s_4(s_3s_2^{-1}s_3)s_4A_4^\times + A_5^{(2)}$. We now prove (4). We have $s_4w^-s_4w^+s_4^{-1} = s_3^{-1}s_4^{-1}s_3(s_2s_1^{-1}s_2)s_4s_3^{-1}s_4s_3s_4^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4s_3$.

Now $s_4s_3^{-1}(s_4s_3s_4^{-1}) = s_4s_3^{-1}s_3^{-1}s_4s_3) = s_4s_3^{-2}s_4s_3 \in R^\times s_4s_3s_4s_3 + u_4s_3 + Rs_4s_3^{-1}s_4s_3 = R^\times(s_4s_3s_4)s_3 + u_4s_3 + Rs_4s_3^{-1}s_4s_3 = R^\times s_3s_4s_3^2 + u_4s_3 + Rs_4s_3^{-1}s_4s_3 \subset R^\times s_3s_4s_3^{-1} + Rs_3s_4s_3 + Rs_3s_4 + u_4s_3 + Rs_4s_3^{-1}s_4s_3$. Moreover, by Lemma 6.11(2), 6.11(1), 6.8(4) and Lemma 6.11, respectively, we have

$$\begin{aligned}
&s_4^{-1}s_3(s_2s_1^{-1}s_2)(u_4s_3)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 && \subset A_5^{(2)} \\
&s_4^{-1}s_3(s_2s_1^{-1}s_2)(s_3s_4)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 && \in A_5^{(2)} \\
&s_4^{-1}s_3(s_2s_1^{-1}s_2)(s_3s_4s_3)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 && \in A_5^{(2)} \\
&s_4^{-1}s_3(s_2s_1^{-1}s_2)(s_4s_3^{-1}s_4s_3)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 && = s_4^{-1}s_3s_4^2(s_2s_1^{-1}s_2)s_3(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1}s_3^{-1} \in A_5^{(2)}
\end{aligned}$$

It follows that $s_4w^-s_4w^+s_4^{-1} \in u_3^\times s_4^{-1}s_3(s_2s_1^{-1}s_2)s_3s_4s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4u_3^\times + A_5^{(2)}$ hence $s_4w^-s_4w^+s_4^{-1} \in u_3^\times s_4^{-1}w^+ s_4w^-s_4u_3^\times + A_5^{(2)}$.

The proof of (5) is similar: one first gets $s_4w^-s_4^{-1}w^+s_4^- = s_3^{-1}s_4^{-1}s_3(s_2s_1^{-1}s_2)s_4s_3^{-1}s_4^{-1}s_3s_4^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4s_3$ and then writes down $(s_4s_3^{-1}s_4^{-1})s_3s_4^{-1} = s_3^{-1}s_4^{-1}s_3^2s_4^{-1} \in R^\times s_3^{-1}(s_4^{-1}s_3^{-1}s_4^{-1}) + R(s_3^{-1}s_4^{-1}s_3)s_4^{-1} + Rs_3^{-1}s_4^{-2} = R^\times s_3^{-2}s_4^{-1}s_3^{-1} + Rs_4s_3^{-1}s_4^{-2} + Rs_3^{-1}s_4^{-2} \subset R^\times s_3s_4^{-1}s_3^{-1} + Rs_3^{-1}s_4^{-1}s_3^{-1} + Rs_4^{-1}s_3^{-1} + Rs_4s_3^{-1}u_4 + Rs_3^{-1}s_4^{-2}$; one then shows using the same arguments as before that all terms but $R^\times s_3s_4^{-1}s_3^{-1}$ provide an element of $A_5^{(2)}$, thus $s_4w^-s_4^{-1}w^+s_4^- \in u_3^\times s_4^{-1}w^+s_4^{-1}w^-s_4u_3^\times + A_5^{(2)}$.

We prove (6).

$$\begin{aligned}
s_4s_3A_3s_3^{-1}s_4s_3^{-1}A_3s_3s_4 &\subset s_4s_3A_3s_3^{-1}s_4s_3^{-1}A_3(s_3s_4s_3)s_3^{-1} \\
&\subset s_4s_3A_3s_3^{-1}s_4s_3^{-1}A_3s_4s_3s_4s_3^{-1} \\
&\subset s_4s_3A_3(s_3^{-1}s_4s_3^{-1}s_4)A_3s_3s_4s_3^{-1} \\
&\subset s_4s_3A_3s_4^{-1}s_3s_4^{-1}s_3A_3s_3s_4s_3^{-1} + A_5^{(2)} \quad \text{(Lemmas 3.5 and 6.11(1))} \\
&\subset (s_4s_3s_4^{-1})A_3s_3s_4^{-1}s_3A_3s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset s_3^{-1}s_4s_3A_3s_3s_4^{-1}s_3A_3s_3s_4s_3^{-1} + A_5^{(2)} \\
&\subset u_3s_4w^+s_4^{-1}w^+s_4u_3 + A_5^{(2)} \quad \text{(Lemma 6.7)} \quad \square
\end{aligned}$$

Proposition 6.13.

$$u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4 \subset A_4s_4w^-s_4w^-s_4A_4 + A_4s_4w^+s_4^{-1}w^+s_4A_4 + A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4$$
$$+A_4s_4w^-s_4w^+s_4^{-1}A_4 + A_4s_4w^-s_4^{-1}w^+s_4^{-1}A_4 + A_5^{(2)}$$

Proof. We first note that, by Lemma 6.10 and Lemma 6.12(4) and (5), the right-hand side (RHS) of the statement is invariant under Φ and Ψ. We now consider an expression of the form $s_4^{?}s_3^{\alpha}u_2u_1u_2s_3^{\beta}s_4^{?}s_3^{\gamma}u_2u_1u_2s_3^{\delta}s_4^{?}$ with $\alpha,\beta,\gamma \in \{-1,1\}$. By Lemma 6.8 we can assume $\alpha=\beta$ and$\gamma=\delta$, except for the expression $s_4^{\varepsilon}s_3^{\alpha}u_2u_1u_2s_3^{-\alpha}s_4^{\varepsilon}s_3^{-\alpha}u_2u_1u_2s_3^{\alpha}s_4^{\varepsilon}$. Up to applying Φ, we can moreover assume $\varepsilon=1$, and we get the conclusion by Lemma 6.12(6) for $\alpha=1$, by Lemma 6.12(1) and (4) for $\alpha=-1$.

We can now assume $\alpha=\beta$, $\gamma=\delta$, and still $\varepsilon=1$. By Lemma 6.7 this reduces our examination to expressions $x=s_3w^{\alpha}s_4^{\varepsilon}w^{\beta}s_4^{\eta}$ for new parameters $\alpha,\varepsilon,\beta,\eta\in\{-1,1\}$. If $\alpha=\beta=\varepsilon$, we have $x\in A_5^{(2)}$ by Lemma 6.11(6); if $\alpha=\beta=-\varepsilon$, we get $x\in A_5^{(2)}$ if in addition $\eta=-1$, by Lemma 6.11(7) and (8), and $x=s_4w^{\alpha}s_4^{-\alpha}w^{\alpha}s_4\in RHS$ otherwise. As a consequence, we can reduce to the case $\alpha=-\beta$, that is $x=s_4w^{\alpha}s_4^{\varepsilon}w^{-\alpha}s_4^{\eta}$. If $\alpha=1$, $x\in A_5^{(2)}$ by Lemma 6.11(4). If $\alpha=-1$, all the possibilities for x clearly lie in the RHS, except for $s_4w^-s_4^{-1}w^+s_4$, which belongs to $A_5^{(2)}$ by Lemma 6.11(3). This concludes the proof. □

6.3. *Image of the center of the braid group in $A_5^{(3)}/A_5^{(2)}$*

Recall that the center of the braid group B_n is infinite cyclic, generated for $n\geq 3$ by $c_n=(s_1\dots s_{n-1})^n$, and that this generator can be written as $c_n=c_{n-1}y_n=y_nc_{n-1}=y_ny_{n-1}\dots y_3y_2$ where the $y_n\in B_n\setminus B_{n-1}$ under the usual inclusions $B_2\subset B_3\subset\cdots\subset B_{n-1}$ form another family of commuting elements defined by $y_2=s_1^2$ and $y_{n+1}=s_ny_ns_n=s_ns_{n-1}\dots s_2s_1^2s_2\dots s_{n-1}s_n$.

We let $c=c_5=(s_1s_2s_3s_4)^5=(s_4s_3s_2s_1)^5$. The center of G_{32} is cyclic of order 6 and is generated by the image of c. We let $w_0=y_4=s_3s_2s_1^2s_2s_3=c_4c_3^{-1}$, which by definition commutes with B_3, and $\delta=y_5=s_4s_3s_2s_1^2s_2s_3s_4=c_5c_4^{-1}$ which commutes with B_4.

We first need a preparatory lemma.

Lemma 6.14. (1) *In A_4, $s_4^{\alpha}w_0^2s_4^{\beta}w_0s_4^{\gamma}\in A_3^{\times}s_4^{\alpha}w_0^{-1}s_4^{\beta}w_0s_4^{\gamma}+A_3s_4^{\alpha}w_0s_4^{\beta}w_0s_4^{\gamma}+A_5^{(2)}$*
(2) *For all $\alpha,\beta,\gamma,\delta,\varepsilon\in\{-1,1\}=\{-,+\}$,*

$$\begin{aligned}
s_4^{\alpha}w^{\beta}s_4^{\gamma}w^{\delta}s_4^{\varepsilon} &\in s_4^{\alpha}w^{\beta}s_4^{\gamma}w_0^{\delta}s_4^{\varepsilon}A_3^{\times}+s_4^{\alpha}w^{\beta}s_4^{\gamma}u_1u_3u_2u_3s_4^{\varepsilon}A_3\\
&\subset s_4^{\alpha}w^{\beta}s_4^{\gamma}w_0^{\delta}s_4^{\varepsilon}A_3^{\times}+A_5^{(2)}\\
s_4^{\alpha}w^{\beta}s_4^{\gamma}w^{\delta}s_4^{\varepsilon} &\in A_3^{\times}s_4^{\alpha}w_0^{\beta}s_4^{\gamma}w^{\delta}s_4^{\varepsilon}+A_3s_4^{\alpha}u_3u_2u_3u_1s_4^{\gamma}w^{\delta}s_4^{\varepsilon}\\
&\subset A_3^{\times}s_4^{\alpha}w_0^{\beta}s_4^{\gamma}w^{\delta}s_4^{\varepsilon}+A_5^{(2)}\\
s_4^{\alpha}w^{\beta}s_4^{\gamma}w^{\delta}s_4^{\varepsilon} &\in A_3^{\times}s_4^{\alpha}w_0^{\beta}s_4^{\gamma}w_0^{\delta}s_4^{\varepsilon}+A_3s_4^{\alpha}u_3u_2u_3u_1s_4^{\gamma}w^{\delta}s_4^{\varepsilon}+A_3s_4^{\alpha}w_0^{\alpha}s_4^{\gamma}u_3u_2u_3u_1s_4^{\varepsilon}\\
&\subset A_3^{\times}s_4^{\alpha}w_0^{\beta}s_4^{\gamma}w_0^{\delta}s_4^{\varepsilon}+A_5^{(2)}
\end{aligned}$$

Proof. (1) is a straightforward consequence of Lemma 4.9 and of the fact that $s_4^{\alpha}U_0s_4^{\beta}w_0s_4^{\gamma}\subset s_4^{\alpha}A_3u_3A_3s_4^{\beta}w_0s_4^{\gamma}+s_4^{\alpha}A_3u_3u_2u_3s_4^{\beta}w_0s_4^{\gamma}=A_3s_4^{\alpha}u_3s_4^{\beta}w_0s_4^{\gamma}A_3+A_3s_4^{\alpha}u_3u_2u_3s_4^{\beta}w_0s_4^{\gamma}A_3\subset A_5^{(2)}$ by Proposition 6.3. (2) follows from an easy variation in the proof of Lemma 6.7 and from Lemma 4.6. □

We are then in position to prove the following.

Lemma 6.15. (1) $s_4w^-s_4w^+s_4^{-1}\in A_4^{\times}s_4w^+s_4^{-1}w^+s_4+A_5^{(2)}$
(2) $s_4w^-s_4^{-1}w^+s_4^{-1}\in s_4^{-1}w^-s_4w^-s_4^{-1}A_4^{\times}+A_5^{(2)}$.

Proof. We have $s_4w^-s_4w^+s_4^{-1}\in A_3^{\times}s_4w_0^{-1}s_4w_0s_4^{-1}+A_5^{(2)}$ by Lemma 6.14(2). Since $s_4w_0^{-1}s_4w_0s_4\in A_3^{\times}s_4w^-s_4w^+s_4\subset A_5^{(2)}$ by Lemma 6.11 (5) and since $s_4^{-1}\in R^{\times}s_4^2+Rs_4+R$, we have $s_4w_0^{-1}s_4w_0s_4^{-1}\equiv s_4w_0^{-1}s_4w_0s_4^2 \bmod A_5^{(2)}$. Then $s_4w_0^{-1}s_4w_0s_4^2\in A_3^{\times}s_4w_0^2s_4w_0s_4^2+A_3s_4w_0s_4w_0s_4^2+A_5^{(2)}$ by Lemma 6.14(1), and $s_4w_0s_4w_0s_4^2\in Rs_4w_0s_4w_0s_4+Rs_4w_0s_4w_0s_4^{-1}+A_5^{(2)}$. Then

$$s_4w_0s_4w_0s_4^{-1}\in A_3^{\times}s_4w^+s_4w^+s_4^{-1}\subset A_5^{(2)}\quad\text{and}\quad s_4w_0s_4w_0s_4\in A_3^{\times}s_4w^+s_4w^+s_4\subset A_5^{(2)}$$

by Lemmas 6.14(2) and 6.11(6). It follows that $s_4w_0^{-1}s_4w_0s_4^2\in A_3^{\times}s_4w_0^2s_4w_0s_4^2+A_5^{(2)}$.

Now $s_4w_0^2s_4w_0s_4^2=s_4w_0(w_0(s_4w_0s_4))s_4$ and $w_0(s_4w_0s_4)=c_3^{-1}c_5\in A_3^{\times}c_5$ commutes with w_0 and s_4. Thus $s_4w_0^2s_4w_0s_4^2=(w_0(s_4w_0s_4))s_4w_0s_4\in A_4^{\times}s_4w_0s_4^2w_0s_4$. Now $s_4w_0s_4^2w_0s_4\in R^{\times}s_4w_0s_4^{-1}w_0s_4+Rs_4w_0s_4w_0s_4+A_5^{(2)}$; moreover we already noticed $s_4w_0s_4w_0s_4\in A_5^{(2)}$, hence $s_4w_0s_4^2w_0s_4\in R^{\times}s_4w_0s_4^{-1}w_0s_4+A_5^{(2)}\subset A_3^{\times}s_4w^+s_4^{-1}w^+s_4+A_5^{(2)}$ by Lemma 6.14(2) , and this proves (1).

Now we have $s_4w^-s_4^{-1}w^+s_4^{-1}=\Psi(s_4w^-s_4w^+s_4^{-1})\in\Psi(A_4^{\times}s_4w^+s_4^{-1}w^+s_4)+\Psi(A_5^{(2)})=s_4^{-1}w^-s_4w^-s_4^{-1}A_4^{\times}+A_5^{(2)}$, and this proves (2). □

By a direct computation, we will prove the following lemma, which will turn out to be crucial in the proof of the main theorem. We postpone this (lengthy) calculation to Section 7.

Lemma 6.16. *In A_5, δ^3 belongs to*

$$A_4^{\times}s_4w^-s_4w^-s_4A_3^{\times}+A_4s_4w^+s_4^{-1}w^+s_4A_4+A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_5^{(2)}$$

6.4. Right actions are left actions

Lemma 6.17. (1) *For all $\alpha,\beta,\gamma,\ldots\in\{-1,1\}$, $x,y\in A_3$, $s_1(s_4^{\alpha}s_3^{\beta}xs_3^{\gamma}s_4^{\delta}s_3^{\varepsilon}ys_3^{\zeta}s_4^{\eta})\equiv(s_4^{\alpha}s_3^{\beta}xs_3^{\gamma}s_4^{\delta}s_3^{\varepsilon}ys_3^{\zeta}s_4^{\eta})s_1 \bmod A_5^{(2)}$.*
(2) *For all $x\in A_4$, $(s_4w^+s_4^{-1}w^+s_4)x\in A_4(s_4w^+s_4^{-1}w^+s_4) \bmod A_5^{(2)}$*
(3) *For all $x\in A_4$, $(s_4^{-1}w^-s_4w^-s_4^{-1})x\in A_4(s_4^{-1}w^-s_4w^-s_4^{-1}) \bmod A_5^{(2)}$*
(4) $(s_4w^-s_4w^+s_4^{-1})s_3^{-1}\in s_3^{-1}(s_4w^-s_4w^+s_4^{-1})+A_5^{(2)}$
(5) $(s_4w^-s_4^{-1}w^+s_4^{-1})s_3^{-1}\in u_3s_4^{-1}w^+s_4^{-1}w^-s_4+A_5^{(2)}$.

Proof. We first prove (1). By Lemma 6.7 and because s_1 commutes with u_1 we can assume $x=y=s_2^{-1}s_1s_2^{-1}$. Since $(s_2^{-1}s_1s_2^{-1})s_1\in s_1(s_2^{-1}s_1s_2^{-1})+u_1u_2u_1$ by Lemma 2.3 (and even $(s_2^{-1}s_1s_2^{-1})s_1\in s_1(s_2^{-1}s_1s_2^{-1})+u_1u_2+u_2u_1$, see Lemma 3.5), by Proposition 6.3 we get the conclusion.

We then prove (2). Because of (1), and because we have the result for $x=s_3^{-1}$ by Lemma 6.12(2), we need only consider $x=s_2$. For $x=s_2$, we first use that $s_4w^+s_4^{-1}w^+s_4$ belongs to $u_1^{\times}s_4s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}\,s_2s_3s_4u_1^{\times}+A_5^{(2)}$ by Lemma 6.7; then, because of (1) we get that $u_1^{\times}s_4s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}\,s_2s_3s_4u_1^{\times}$ belongs to $u_1^{\times}s_4s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}\,s_2s_3s_4+A_5^{(2)}$. Then

$$\begin{aligned}s_4w^+s_4^{-1}w^+s_4.s_2 &\in u_1^{\times}s_4(s_3s_2s_1^{-1}s_2s_3)s_4^{-1}(s_3s_2s_1^{-1}\,s_2s_3)s_4s_2+A_5^{(2)}\\ &\subset u_1^{\times}s_2s_4(s_3s_2s_1^{-1}s_2s_3)s_4^{-1}(s_3s_2s_1^{-1}\,s_2s_3)s_4+A_5^{(2)}\end{aligned}$$

because s_2 commutes with both s_4 and $s_3s_2s_1^{-1}s_2s_3$ and this proves (2). One gets (3) by applying Φ to (2).

We prove (4). One easily gets $(s_4w^-s_4w^+s_4^{-1})s_3^{-1}=s_4s_3^{-1}s_2s_1^{-1}s_2s_4s_3s_4^{-2}s_2s_1^{-1}s_2s_3^{-1}s_4$. Now $s_4^{-2}\in R^{\times}s_4+Rs_4^{-1}+R$, and it is easily checked that the terms originating from Rs_4^{-1} and R belong to $A_5^{(2)}$. We thus get $s_4s_3^{-1}s_2s_1^{-1}s_2s_4s_3s_4^{-2}s_2s_1^{-1}s_2s_3^{-1}s_4\in s_4s_3^{-1}s_2s_1^{-1}s_2(s_4s_3s_4)s_2s_1^{-1}s_2s_3^{-1}s_4+A_5^{(2)}\subset s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4+A_5^{(2)}$, and $s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4$ belongs to $u_3s_4^-w^+s_4w^-s_4+A_5^{(2)}$ by Lemma 6.12(1). We prove (5). One easily gets $(s_4w^-s_4^{-1}w^+s_4^{-1})s_3^{-1}=s_4s_3^{-1}s_2s_1^{-1}s_2s_4s_3^{-1}s_4^{-2}s_2s_1^{-1}s_2s_3^{-1}s_4$, and $s_4s_3^{-1}s_2s_1^{-1}s_2s_4s_3^{-1}xs_2s_1^{-1}s_2s_3^{-1}s_4\in A_5^{(2)}$ for $x\in 1,s_4^{-1}$, hence $(s_4w^-s_4^{-1}w^+s_4^{-1})s_3^{-1}$ belongs to

$$\begin{aligned}&s_4s_3^{-1}s_2s_1^{-1}s_2s_4s_3^{-1}s_4s_2s_1^{-1}s_2s_3^{-1}s_4A_5^{(2)} &&= s_4s_3^{-1}s_4s_2s_1^{-1}s_2s_3^{-1}s_4s_2s_1^{-1}s_2s_3^{-1}s_4A_5^{(2)}\\ \subset\ &u_3s_4^{-1}s_3s_4^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4s_2s_1^{-1}s_2s_3^{-1}s_4A_5^{(2)} &&= u_3s_4^{-1}s_3s_2s_1^{-1}s_2(s_4^{-1}s_3^{-1}s_4)s_2s_1^{-1}s_2s_3^{-1}s_4A_5^{(2)}\\ =\ &u_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4A_5^{(2)} &&\subset u_3s_4^{-1}w^+s_4^{-1}w^-s_4+A_5^{(2)}\quad\Box\end{aligned}$$

Remark 6.18. Another proof of item (2). It is easily checked that $s_4w^+s_4^{-1}w^+s_4\equiv(s_4s_3s_2s_1^2s_2s_3s_4)^2 \bmod A_5^{(2)}$, and the element $s_4s_3s_2s_1^2s_2s_3s_4$ of the braid group B_5 is well-known to centralize B_4.

Proposition 6.19.

$$\begin{aligned}A_5^{(3)} &= A_4s_4w^-s_4w^-s_4+A_4s_4w^+s_4^{-1}w^+s_4+A_4s_4^{-1}w^-s_4w^-s_4^{-1}+A_5^{(2)}\\ A_5^{(3)} &= s_4w^-s_4w^-s_4A_4+s_4w^+s_4^{-1}w^+s_4A_4+s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_5^{(2)}\end{aligned}$$

Proof. Clearly the RHS are included in $A_5^{(3)}$. By Propositions 6.6 and 6.13 we have $A_5^{(3)}\subset A_4s_4w^-s_4w^-s_4A_4+A_4s_4w^+s_4^{-1}w^+s_4A_4+A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_4s_4w^-s_4w^+s_4^{-1}A_4+A_4s_4w^-s_4^{-1}w^+s_4^{-1}A_4+A_5^{(2)}$. Lemma 6.15 then implies $A_5^{(3)}\subset A_4s_4w^-s_4w^-s_4A_4+A_4s_4w^+s_4^{-1}w^+s_4A_4+A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_5^{(2)}$. By Lemma 6.17 this implies $A_5^{(3)}\subset A_4s_4w^-s_4w^-s_4A_4+A_4s_4w^+s_4^{-1}w^+s_4+A_4s_4^{-1}w^-s_4w^-s_4^{-1}+A_5^{(2)}$ and $A_5^{(3)}\subset A_4s_4w^-s_4w^-s_4A_4+s_4w^+s_4^{-1}w^+s_4A_4+s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_5^{(2)}$. Now, by Lemma 6.16, $s_4w^-s_4w^-s_4$ belongs to $A_4^{\times}\delta^3A_3^{\times}+A_4s_4w^+s_4^{-1}w^+s_4A_4+A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_5^{(2)}$, hence $A_4s_4w^-s_4w^-s_4A_4\subset A_4\delta^3A_4+A_4s_4w^+s_4^{-1}w^+s_4A_4+A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_5^{(2)}=A_4c^3A_4+A_4s_4w^+s_4^{-1}w^+s_4A_4+A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_5^{(2)}=A_4c^3A_4+A_4s_4w^+s_4^{-1}w^+s_4A_4+s_4^{-1}w^-s_4w^-s_4^{-1}A_4+A_5^{(2)}$ and, since c^3 is central and by Lemma 6.17, this latter expression can be written as $A_4c^3+A_4s_4w^+s_4^{-1}w^+s_4+s_4^{-1}w^-s_4w^-s_4^{-1}+A_5^{(2)}=A_4\delta^3+A_4s_4w^+s_4^{-1}w^+s_4+s_4^{-1}w^-s_4w^-s_4^{-1}+A_5^{(2)}=A_4s_4w^-s_4w^-s_4A_3^{\times}+A_4s_4w^+s_4^{-1}w^+s_4+s_4^{-1}w^-s_4w^-s_4^{-1}+A_5^{(2)}=A_4s_4w^-s_4w^-s_4+A_4s_4w^+s_4^{-1}w^+s_4+s_4^{-1}w^-s_4w^-s_4^{-1}+A_5^{(2)}$. The other expression is deduced from this one by application of $\Phi\circ\Psi$. $\Box$

Proposition 6.20. $A_5=A_5^{(3)}$.

Proof. One only needs to prove $A_5^{(4)} \subset A_5^{(3)}$, that is $u_4au_4bu_4cu_4 \subset A_5^{(3)}$ for all $a, b, c \in A_4$. We have $u_4au_4bu_4c \in A_5^{(3)} = A_5^{(2)}A_4s_4w^-s_4w^-s_4 + A_4s_4w^+s_4^{-1}w^+s_4 + A_4s_4^{-1}w^-s_4w^-s_4^{-1}$ hence

$$u_4au_4bu_4cu_4 \subset A_5^{(2)}u_4A_4s_4w^-s_4w^-s_4u_4 + A_4s_4w^+s_4^{-1}w^+s_4u_4 + A_4s_4^{-1}w^-s_4w^-s_4^{-1}u_4 \subset A_5^{(3)}.$$

This proves the claim. $\square$

This proves Theorem 6.1, and actually the following refinement:

Theorem 6.21.

$$\begin{aligned} A_5 \ = \ & A_4 + A_4s_4A_4 + A_4s_4^{-1}A_4 + A_4s_4s_3^{-1}s_4A_4 + A_4s_4^{-1}s_3s_2^{-1}s_3s_4^{-1}A_4 + A_4s_4s_3^{-1}s_2s_3^{-1}s_4A_4 \\ & + A_4s_4^{-1}w^+s_4^{-1}A_4 + A_4s_4w^-s_4A_4 + A_4s_4^{-1}w^-s_4^{-1}A_4 + A_4s_4w^+s_4A_4 + s_4w^-s_4w^-s_4A_4 \\ & + s_4w^+s_4^{-1}w^+s_4A_4 + s_4^{-1}w^-s_4w^-s_4^{-1}A_4 \end{aligned}$$

6.5. A_5 as an A_4-module

We need the following lemma on A_3:

Lemma 6.22. (1) $u_2u_1u_2 \subset u_1s_2s_1^2s_2 + u_1u_2u_1$
(2) $u_2u_1u_2 \subset u_1s_2^{-1}s_1^{-2}s_2^{-1} + u_1u_2u_1$.

Proof. (2) is a consequence of (1) by using Φ, so it is enough to prove (1). We have $u_2u_1u_2 \subset u_1u_2u_1 + \sum_{\alpha\in\{-1,1\}} Rs_2^{\alpha}s_1^{-\alpha}s_2^{\alpha}$ because u_i is R-spanned by $1, s_i, s_i^{-1}$ and because of Lemma 2.2. Moreover $s_2^{-1}s_1s_2^{-1} \in u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ by Lemmas 2.4 and 2.3, hence $u_2u_1u_2 \subset u_1s_2s_1^{-1}s_2 + u_1u_2u_1$. Since $s_1^{-1} \in Rs_1^2 + Rs_1 + R$ we get $s_2s_1^{-1}s_2 \in Rs_2s_1^2s_2 + Rs_2s_1s_2 + Rs_2^2 \subset Rs_2s_1^2s_2 + u_1u_2u_1$ hence $u_2u_1u_2 \subset u_1s_2s_1^2s_2 + u_1u_2u_1$. $\square$

We introduce or re-introduce the following submodules of A_5:

$$\begin{aligned} A_5^{(1)} &= A_4u_4A_4 = A_4 + A_4s_4A_4 + A_4s_4^{-1}A_4 \\ A_5^{(1\frac{1}{4})} &= A_5^{(1)} + A_4s_4s_3^{-1}s_4A_4 (= A_5^{(1)} + A_4sh^2(A_3)A_4) \\ A_5^{(1\frac{1}{2})} &= A_5^{(1\frac{1}{4})} + A_4u_4u_3u_2u_3u_4A_4 = A_5^{(1\frac{1}{4})} + A_4s_4s_3^{-1}s_2s_3^{-1}s_4A_4 + A_4s_4^{-1}s_3s_2^{-1}s_3s_4^{-1}A_4 \\ A_5^{(2)} &= A_4u_4A_4u_4A_4 = A_5^{(1\frac{1}{2})} + \textstyle\sum_{\alpha,\beta\in\{1,-1\}} A_4s_4^{\alpha}w^{\beta}s_4^{\alpha}A_4 \end{aligned}$$

and

$$A_5 = A_5^{(3)} = A_4u_4A_4u_4A_4u_4A_4 = A_5^{(2)} + A_4s_4w^-s_4w^-s_4 + A_4s_4w^+s_4^{-1}w^+s_4 + A_4s_4^{-1}w^-s_4w^-s_4^{-1}$$

We let $\mathcal{B}$ denote the family of elements defined in Corollary 5.12, which span A_4 as a left B-module, $\mathcal{A}$ the family spanning A_4 as an A_3-module defined in Proposition 4.8, and $\mathcal{A}'$ its image under the automorphism Ad Δ of A_4 (that is $s_1 \leftrightarrow s_3, s_2 \leftrightarrow s_2$). We prove the following.

Lemma 6.23. (1) $A_5^{(1)} = A_4 + \sum_{x\in\mathcal{A}} A_4s_4x + \sum_{x\in\mathcal{A}} A_4s_4^{-1}x$
(2) $A_5^{(1\frac{1}{4})} = A_5^{(1)} + \sum_{x\in\mathcal{B}} A_4s_4s_3^{-1}s_4x$.

Proof. (1) is a consequence of $A_3s_4^{\pm1} = s_4^{\pm1}A_3$, because

$$A_5^{(1)} = A_4 + A_4s_4A_4 + A_4s_4^{-1}A_4 = A_4 + A_4s_4\sum_{x\in\mathcal{A}} A_3x + A_4s_4^{-1}\sum_{x\in\mathcal{A}} A_3x = A_4 + \sum_{x\in\mathcal{A}} A_4s_4x + \sum_{x\in\mathcal{A}} A_4s_4^{-1}x$$

We prove (2). We have $(s_4s_3^{-1}s_4)s_1 = s_1(s_4s_3^{-1}s_4)$ and $(s_4s_3^{-1}s_4)s_3^{-1} \in s_3^{-1}(s_4s_3^{-1}s_4) + u_3u_4 + u_4u_3$ by Lemma 3.5, hence $(s_4s_3^{-1}s_4)B \subset B(s_4s_3^{-1}s_4) + A_5^{(1)}$, where we recall $B = \langle s_1, s_3^{-1}\rangle = \langle s_1, s_3\rangle$. Thus

$$A_5^{(1\frac{1}{4})} = A_5^{(1)} + A_4s_4s_3^{-1}s_4\sum_{x\in\mathcal{B}} Bx = A_5^{(1)} + \sum_{x\in\mathcal{B}} A_4s_4s_3^{-1}s_4Bx = A_5^{(1)} + \sum_{x\in\mathcal{B}} A_4s_4s_3^{-1}s_4x \quad \square$$

Lemma 6.24. (1) $A_5^{(1\frac{1}{2})} \subset A_5^{(1\frac{1}{4})} + A_4s_4s_3s_2^2s_3s_4A_4 + A_4s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1}A_4$
(2) $A_5^{(1\frac{1}{2})} \subset A_5^{(1\frac{1}{4})} + \sum_{x\in\mathcal{A}'} A_4s_4s_3s_2^2s_3s_4x + \sum_{x\in\mathcal{A}'} A_4s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1}x$.

Proof. We have $s_3^{-1}s_2s_3^{-1} \subset u_2s_3s_2^2s_3 + u_2u_3u_2$ by Lemma 6.22 hence $s_4s_3^{-1}s_2s_3^{-1}s_4 \subset s_4u_2s_3s_2^2s_3s_4 + s_4u_2u_3u_2s_4 \subset u_2s_4s_3s_2^2s_3s_4 + u_2s_4u_3s_4u_2 \subset u_2s_4s_3s_2^2s_3s_4 + A_5^{(1\frac{1}{4})}$. Applying Φ this implies $s_4^{-1}s_3s_2^{-1}s_3s_4^{-1} \subset u_2s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1} + A_5^{(1\frac{1}{4})}$ which proves (1). Let $A_3' = \langle s_2, s_3\rangle$. We have $A_4 = \sum_{x\in\mathcal{A}'} x$. Since $s_4s_3s_2^2s_3s_4$ commutes with s_2 and s_3 hence to A_3', we get

$$A_4s_4s_3s_2^2s_3s_4A_4 \subset \sum_{x\in\mathcal{A}'} A_4s_4s_3s_2^2s_3s_4A_3'x \subset \sum_{x\in\mathcal{A}'} A_4s_4s_3s_2^2s_3s_4x$$

and similarly $s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1} = (s_4s_3s_2^2s_3s_4)^{-1}$ commutes with s_2 and s_3 hence

$$A_4s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1}A_4 \subset \sum_{x\in\mathcal{A}'} A_4s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1}x$$

which proves (2). $\square$

Lemma 6.25.

$$A_5^{(2)} = A_5^{(1\frac{1}{2})} + \sum_{\alpha\in\{-1,1\}} A_4s_4^{\alpha}w^{\alpha}s_4^{\alpha} + \sum_{\alpha\in\{-1,1\}}\sum_{x\in\mathcal{A}} A_4s_4^{\alpha}w^{-\alpha}s_4^{\alpha}x$$

Proof. By Lemma 4.6(1), we have $w_0 \in A_3^{\times}w^+ + U_0$, $w_0^{-1} \in A_3^{\times}w^- + U_0$, hence $w^+ \in A_3^{\times}w_0 + U_0$, $w^- \in A_3^{\times}w_0^{-1} + U_0$, with $U_0 = A_3u_3A_3 + A_3u_3u_2u_3A_3 \subset A_4$. As a consequence, for $\alpha, \beta \in \{-1, 1\}$, we have $A_4s_4^{\alpha}w^{\beta}s_4^{\alpha}A_4 \subset A_4s_4^{\alpha}A_3^{\times}w_0^{\beta}s_4^{\alpha}A_4 + A_4s_4^{\alpha}U_0s_4^{\alpha}A_4$ Moreover, $s_4^{\alpha}U_0s_4^{\alpha} = s_4^{\alpha}A_3u_3A_3s_4^{\alpha} + s_4^{\alpha}A_3u_3u_2u_3A_3s_4^{\alpha} = A_3s_4^{\alpha}u_3s_4^{\alpha}A_3 + A_3s_4^{\alpha}u_3u_2u_3s_4^{\alpha}A_3 \subset A_5^{(1\frac{1}{4})} + A_5^{(1\frac{1}{2})} = A_5^{(1\frac{1}{2})}$, hence $A_4s_4^{\alpha}w^{\beta}s_4^{\alpha}A_4 \subset A_4s_4^{\alpha}w_0^{\beta}s_4^{\alpha}A_4 + A_5^{(1\frac{1}{2})}$. Since w_0 and s_4 commute with s_1 and s_2, we have

$$A_4s_4^{\alpha}w_0^{\beta}s_4^{\alpha}A_4 \subset \sum_{x\in\mathcal{A}} A_4s_4^{\alpha}w_0^{\beta}s_4^{\alpha}A_3x \subset \sum_{x\in\mathcal{A}} A_4s_4^{\alpha}w_0^{\beta}s_4^{\alpha}x$$

If moreover $\alpha = \beta$, $s_4^{\alpha}w_0^{\alpha}s_4^{\alpha} = (s_4s_3s_2s_1^2s_2s_3s_4)^{\alpha}$ commutes with $\langle s_1, s_2, s_3\rangle = A_4$, hence $A_4s_4^{\alpha}w_0^{\alpha}s_4^{\alpha}A_4 = A_4s_4^{\alpha}w_0^{\alpha}s_4^{\alpha}$, and this concludes the proof. $\square$

From this one can conclude the following.

Theorem 6.26. (1) $A_5 = A_5^{(3)}$ *is generated as an* A_4*-module by* 240 *elements.*
(2) $A_5 = A_5^{(3)}$ *is generated as an* R*-module by* 155,520 *elements.*

Proof. By Lemma 6.22, $A_5^{(1)}$ is generated as an A_4-module by $1 + 2 \times 27 = 55$ elements, $A_5^{(1\frac{1}{4})}$ by $A_5^{(1)}$ and $|\mathcal{B}| = 72$ elements, $A_5^{(1\frac{1}{2})}$ after Lemma 6.24 by $A_5^{(1\frac{1}{4})}$ and $2 \times |\mathcal{A}'| = 2 \times 27 = 54$ elements, $A_5^{(2)}$ by $A_5^{(1\frac{1}{2})}$ and $2 + 2 \times |\mathcal{A}| = 56$ elements (Lemma 6.25), and $A_5^{(3)}$ by $A_5^{(2)}$ and 3 elements. It follows that A_5 is A_4-generated by $55 + 72 + 54 + 56 + 3 = 240$ elements, which proves (1). Since A_4 is R-generated by 648 elements, we get that A_5 is R-generated by $240 \times 648 = 155{,}520$ elements, which proves (2). $\square$

7. Proof of Lemma 6.16

For the sake of concision we denote $V_0 = A_5^{(2)}$ and $V^+ = A_5^{(2)} + A_4s_4w^+s_4^{-1}w^+s_4A_4 = V_0 + A_4s_4w^+s_4^{-1}w^+s_4A_4$. We will prove that $X \in A_4^{\times}s_4w^-s_4w^-s_4A_4^{\times} + V^+$, starting from $X = \delta^3 = s_4w_0s_4^2w_0s_4^{-1}w_0s_4$ to $X = \delta^3 = s_4w^-s_4w^-s_4$ (for which the statement is trivial) through a sequence of reductions of the type $X \to X'$ where $X' \in A_4^{\times}XA_3^{\times} + V^+$. For publication purposes, the proof of the intermediate lemmas are skipped. We only indicate which ones of the previous lemmas are used in the proof. The interested reader can refer to the lengthier version on the arxiv for more details.

7.1. *Reduction to* $s_4w_0s_4^2w_0s_4^{-1}w_0s_4$

Using $s_4^2 \in R^{\times}s_4^{-1} + Rs_4 + R$ we get $s_4w_0s_4^2w_0s_4^2w_0s_4 \in R^{\times}s_4w_0s_4^2w_0s_4^{-1}w_0s_4 + Rs_4w_0s_4^2w_0s_4w_0s_4 + Rs_4w_0s_4^2w_0^2s_4$. The fact that $s_4w_0s_4^2w_0s_4w_0s_4$, $s_4w_0s_4^2w_0^2s_4$ belongs to V^+ is proved in the following Lemma 7.1, whose proof uses Lemmas 4.9, 6.11(5) (6) (7) (8) and 6.12(4).

Lemma 7.1. (1) $s_4w_0s_4^2w_0^2s_4 \in V^+$.
(2) $s_4w_0s_4^2w_0s_4w_0s_4 \in V^+$
(3) $s_4w_0^2s_4^{-1}w_0s_4 \in V^+$
(4) $s_4w_0s_4w_0s_4^{-1}w_0s_4 \in V_0$.

7.2. Reduction to $s_4 s_3(s_2 s_1^2 s_2)s_3^{-1} s_4 s_3^2 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$

Using $s_4^2 \in R^\times s_4^{-1} + Rs_4 + R$ we get $s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4 \in R^\times s_4 w_0 s_4^{-1} w_0 s_4^{-1} w_0 s_4 + Rs_4 w_0^2 s_4^{-1} w_0 s_4 + Rs_4 w_0 s_4 w_0 s_4^{-1} w_0 s_4$. The fact that $Rs_4 w_0^2 s_4^{-1} w_0 s_4 + Rs_4 w_0 s_4 w_0 s_4^{-1} w_0 s_4 \subset V^+$ has been proved in Lemma 7.1. Finally, $s_4 w_0 s_4^{-1} w_0 s_4^{-1} w_0 s_4 = s_3^{-1}.s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^2 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$ is easily checked to hold in the braid group B_5. Before going further, we first need to establish several lemmas. For proving the first one, one uses Proposition 6.3.

Lemma 7.2. (1) *For all $\alpha, \beta \in \mathbf{Z}$, $s_4 w_0 s_4^\alpha s_3^\beta w_0 s_4 \in V^+$.*
(2) *For all $\alpha, \beta \in \mathbf{Z}$, $s_4 w_0 s_3^\beta s_4^\alpha w_0 s_4 \in V^+$.*

From this lemma the next one follows.

Lemma 7.3. (1) $s_4 w_0 s_4^{-1} s_3 s_4^{-1} w_0 s_4 \in V^+$
(2) $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 w_0 s_4^{-1} w_0 s_4 \in V^+$.

The proof of the next lemma uses Theorem 4.1, Proposition 6.3 and Lemmas 6.11(2) (5), 6.8(3) (4) (6), 3.5, 6.7 and 6.8.

Lemma 7.4. (1) *For all β, $s_4 A_4 s_4 s_3^\beta w_0 s_4 \subset V^+$*
(2) $s_4 s_3 s_2^{-1} s_3 u_1 u_2 s_4 u_3 w_0 s_4 \subset V^+$
(3) $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \in V^+$
(4) $u_4 u_3 u_2 u_3 u_4 A_4 s_4 \subset V^+$*; moreover* $s_4^\alpha u_3 u_2 u_3 s_4^\beta A_4 s_4 \subset V_0$ *when $\alpha, \beta \in \{-1, 1\}$ with $(\alpha, \beta) \neq (1, 1)$*
(5) $u_4 U_0 u_4 A_4 s_4 \subset V^+$
(6) $u_4 s_3^{\pm} A_3 s_3^{\mp} u_4 A_4 s_4 \subset V^+$.

7.3. Reduction to $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$

Expanding s_3^2, we get

$$\begin{aligned} s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^2 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \in\ & R^\times s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ & + Rs_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ & + Rs_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \end{aligned}$$

We have that both $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$ and $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$ belong to V^+ by Lemmas 7.4(3) and 7.3(2). The proof of the next lemma involves Lemmas 6.11(2), 7.4(1).

Lemma 7.5. (1) *For all $\alpha \in \mathbf{Z}$ $s_4 s_3^{-1} s_2^\alpha s_3^{-1} s_4^{-1} s_3 w_0 s_4 \in V^+$*
(2) $s_4 s_3^{-1} s_2^2 s_3^{-1} s_4^2 s_3 w_0 s_4 \in V^+$
(3) $s_4 s_3^{-1} s_2 s_4 s_3^{-1} s_2^2 s_3 s_4^{-1} w_0 s_4 \in V^+$
(4) $s_4 s_3^{-1} s_2 s_4 s_3^{-1} s_2 s_1 s_2 s_3 s_4^{-1} w_0 s_4 \in V^+$.

The proof of the next lemma involves Lemmas 7.4(1) (4), 6.11(1) and 2.4.

Lemma 7.6. (1) $s_4 s_3^{-1} A_3 s_4 u_3 s_4^{-1} w_0 s_4 \subset V^+$
(2) $s_4 s_3^{-1} s_2 s_4 s_3^{-1} u_1 u_2 u_1 s_3 s_4^{-1} w_0 s_4 \subset V^+$
(3) $s_4 s_3^{-1} s_2 s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3 s_4^{-1} w_0 s_4 \in V^+$.

7.4. Reduction to $s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$

Expanding s_1^2, we get

$$\begin{aligned} s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \in\ & R^\times s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ & + Rs_4 s_3 (s_2 s_1 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ & + Rs_4 s_3 s_2^2 s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \end{aligned}$$

Since $s_4 s_3 (s_2 s_1 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 = s_1 s_4 s_3 s_2 s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 s_1$, the latter two terms belong to

$$\begin{aligned} A_2 s_4 (s_3 u_2 s_3^{-1}) s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 A_2 &= A_2 s_4 s_2^{-1} u_3 s_2 s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 A_2 \\ &\subset A_3 s_4 u_3 s_4 s_2 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 A_2 \end{aligned}.$$

We thus only need to prove that the $s_4 s_3^\alpha s_4 s_2 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$ belong to V^+ for $\alpha \in \{-1, 0, 1\}$. When $\alpha = 0$ this is a consequence of Lemma 7.4(6); when $\alpha = 1$ we get $(s_4 s_3 s_4) s_2 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 = s_3 s_4 s_2^{-1} s_3 (s_2^2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$;

since $s_2^2 s_1^2 s_2 \in u_1 s_2^{-1} s_1 s_2^{-1} + u_1 u_2 u_1$ the conclusion follows from Proposition 6.3. When $\alpha = -1$, expanding s_1^2 we only need to consider the $s_4 s_3^{-1} s_4 s_2 s_3^{-1} s_2 s_1^\beta s_2 s_3 s_4^{-1} w_0 s_4$ for $\beta \in \{-1, 0, 1\}$. The case $\beta = -1$ is a consequence of Lemma 7.6 (3), while the other two cases follow from Lemma 7.5(3) and (4). The proof of the next lemma involves Proposition 6.3 and Lemmas 6.8, 6.11 and 6.12.

Lemma 7.7. (1) $s_4 u_3 s_2 u_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 \subset V^+$
(2) $s_4 s_3 s_2 s_1^{-1} u_3 u_2 u_4 s_3 w_0 s_4 \subset V_0$
(3) $s_4 s_3 s_2 s_1^{-1} u_2 u_3 u_4 s_3 w_0 s_4 \subset V^+$.

7.5. Reduction to $s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1}w_0s_4$

Using $s_2s_1^2s_2 \in s_2^{-1}s_1s_2^{-1}u_1^{\times} + u_1u_2u_1$ we get

$$\begin{aligned} s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \in\ & R^{\times}s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}s_2^{-1}s_1s_2^{-1}s_3s_4^{-1}w_0s_4u_1^{\times} \\ & + Rs_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}u_1u_2u_1s_3s_4^{-1}w_0s_4. \end{aligned}$$

We have $s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}u_1u_2u_1s_3s_4^{-1}w_0s_4 = s_4s_3(s_2s_1^{-1}s_2)u_1s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4s_2^{-1}u_1$.

Using $(s_2s_1^{-1}s_2)u_1 \in u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ we get that $s_4s_3(s_2s_1^{-1}s_2)u_1s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4$ belongs to $u_1s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4 + s_4s_3u_1u_2u_1s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4$. Now

$$\begin{aligned} s_4s_3u_1u_2u_1s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4 &= u_1s_4(s_3u_2s_3^{-1})u_1s_2s_3^{-1}u_4s_3w_0s_4 \\ = u_1s_4s_2^{-1}u_3s_2u_1s_2s_3^{-1}u_4s_3w_0s_4 &= u_1s_2^{-1}s_4u_3s_2u_1s_2s_3^{-1}u_4s_3w_0s_4 \subset V^+ \end{aligned}$$

by Lemma 7.7(1), and $s_4s_3s_2s_1^{-1}(s_2s_3^{-1}s_2s_3^{-1})u_4s_3w_0s_4$ belongs to

$$s_4s_3s_2s_1^{-1}s_3^{-1}s_2s_3^{-1}s_2u_4s_3w_0s_4 + s_4s_3s_2s_1^{-1}u_2u_3u_4s_3w_0s_4 + s_4s_3s_2s_1^{-1}u_3u_2u_4s_3w_0s_4$$

by Lemma 3.5. The latter two terms belong to $V+$ by Lemma 7.7(1) and (2), and

$$\begin{aligned} s_4s_3s_2s_1^{-1}s_3^{-1}s_2s_3^{-1}s_2u_4s_3w_0s_4 &= s_4(s_3s_2s_3^{-1})s_1^{-1}s_2s_3^{-1}s_2u_4s_3w_0s_4 \\ = s_4s_2^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_2u_4s_3w_0s_4 &= s_2^{-1}s_4s_3s_2s_1^{-1}s_2s_3^{-1}u_4s_2s_3w_0s_4 \subset V^+ \end{aligned}$$

by Lemma 7.4(6).

The proof of the next lemma involves Proposition 6.3 and Lemmas 7.4(4), 6.8.

Lemma 7.8. $s_4(s_3s_2^{-1}s_3)u_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \subset V^+$.

The proof of the next lemma involves Proposition 6.3 and Lemmas 7.4(4), 6.8, 2.4.

Lemma 7.9. $s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_2u_3s_4^{-1}w_0s_4 \subset V^+$.

The proof of the next lemma involves Proposition 6.3 and Lemmas 6.11(3) (4) (8).

Lemma 7.10. (1) $s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_3u_2s_4^{-1}w_0s_4 \subset V_0$
(2) $u_4u_3u_2u_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \subset V^+$.

Lemma 7.11. $s_4(s_3s_2^{-1}s_3)s_1s_2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in V^+$.

We have $s_4(s_3s_2^{-1}s_3)(s_1s_2s_1)s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 = s_4(s_3s_2^{-1}s_3)s_2s_1s_2s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 = s_4(s_3s_2^{-1}s_3)s_2s_1s_4(s_2s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4$ which belongs to $s_4(s_3s_2^{-1}s_3)s_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 + s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_2u_3s_4^{-1}w_0s_4 + s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_3u_2s_4^{-1}w_0s_4$ by Lemma 3.5. Now

$$s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_2u_3s_4^{-1}w_0s_4 \subset V^+$$

by Lemma 7.9 , while $s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_3u_2s_4^{-1}w_0s_4 \subset V^+$ by Lemma 7.10(1). We are thus reduced to considering

$$s_4(s_3s_2^{-1}s_3)s_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 \in s_4s_2(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 + s_4u_2u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4$$

by Lemma 2.3. We have

$$s_4s_2(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 = s_2s_4(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4s_2 \in V^+$$

by Lemma 7.8. We have $s_4u_2u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 = u_2s_4u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4s_2$ and $s_4u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ is a linear combination of the $s_4s_3^{\alpha}u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ for $\alpha \in \{0, 1, -1\}$. When $\alpha = 0$ we get $s_4u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 = u_2s_1s_4^2s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \subset V^+$ by Proposition 6.3; for $\alpha = 1$ we have $s_4s_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 = s_3s_2^{-1}s_1s_4u_3s_2s_3^{-1}s_4^{-1}w_0s_4s_1 \subset V^+$ by Proposition 6.3; for $\alpha = -1$ we have that $s_4s_3^{-1}u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 = (s_4s_3^{-1}s_4)u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ lies inside $u_3^{\times}s_4^{-1}s_3s_4^{-1}u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 + u_3u_4u_3u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ by Lemma 2.4, and $u_4u_3u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \subset V^+$ by Lemma 7.10(2). Finally, $s_4^{-1}s_3s_4^{-1}u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ is easily shown to be equal to $s_4^{-1}s_3u_2s_3s_4^{-1}s_1s_3^{-1}s_2s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} \subset V^+$ by Lemma 7.4(4).

7.6. Reduction to $s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4$

In the braid group, we check that $s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1}w_0s_4 = s_2^{-1}s_1^{-1}s_4(s_3s_2^{-1}s_3)s_1s_2^2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4s_1^{-1}s_2^{-1}$.

Expanding s_2^2 we get

$$\begin{aligned} s_4(s_3s_2^{-1}s_3)s_1s_2^2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in\ & R^\times s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \\ & + Rs_4(s_3s_2^{-1}s_3)s_1s_2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \\ & + Rs_4(s_3s_2^{-1}s_3)s_1^2s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \end{aligned}$$

We have $s_4(s_3s_2^{-1}s_3)s_1^2s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in V^+$ by Lemma 7.8, and

$$s_4(s_3s_2^{-1}s_3)s_1s_2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in V^+$$

by Lemma 7.11.

The proof of the next lemma involves the identity $s_4s_3s_1^{-1}s_2s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 = s_1^{-1}s_2^{-1}s_3^{-1}s_4\ s_3\ s_2^2s_3s_4^{-1}s_1s_2s_3w_0s_4$ as well as the identity $s_4s_3(s_2^{-1}s_1^{-1}s_2)s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 = s_1s_2^{-1}s_3^{-1}s_4^{-1}\ s_3\ s_2s_2s_1s_2^{-1}s_3s_2s_4^{-1}s_3^2s_2s_1^2s_2s_3s_4$. It also uses Lemmas 7.4(4) and 6.11(8).

Lemma 7.12. (1) $s_4s_3s_1^{-1}s_2s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$
(2) $s_4s_3s_2^{-1}s_1^{-1}s_2s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V_0$.

7.7. Reduction to $s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4$

We have $s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 = s_4s_3s_2^{-2}s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4$ and, expanding s_2^{-2}, we get

$$\begin{aligned} s_4s_3s_2^{-2}s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \in\ & R^\times s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \\ & + Rs_4s_3s_2^{-1}s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \\ & + Rs_4s_3s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \end{aligned}$$

and the last two terms belong to V_0 by Lemma 7.12(1) and (2).

The proof of the next lemma involves the relation $s_4s_3s_2^{-1}s_3s_1s_2s_4^{-1}s_3s_4^{-1} = (s_1s_2s_3s_2^{-1}s_1^{-1})s_4s_3s_2^{-1}s_3s_4^{-2}(s_1s_2)$ as well as Lemmas 3.5 and 7.4(4), 6.11(2).

Lemma 7.13. (1) $s_4(s_3s_2^{-1}s_3)s_1s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$
(2) $s_4(s_3s_2^{-1}s_3)s_1s_2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$
(3) $s_4s_3s_2^{-1}s_3s_1s_2s_4^{-1}s_3s_4^{-1} \in A_4^\times s_4s_3s_2^{-1}s_3s_4^{-2}A_3^\times$
(4) $s_4u_2u_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \in V_0$.

For the next lemma we use $A_4 = A_3u_3A_3 + A_3u_3u_2u_3 + A_3w^+ + A_3w^-$ as well as Proposition 6.3 and Lemmas 6.11(2), 6.8(4), 6.11(7), 6.11(5) , 7.4(4).

Lemma 7.14. (1) $u_4A_4s_4^{-1}s_3^\beta\ w_0s_4 \subset V^+$
(2) $s_4u_3u_2s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \subset V^+$.

7.8. Reduction to $s_4w^-s_4w_0^2s_4$

We have the identity $s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 = (s_2^{-1}s_1^{-1})s_4(s_3s_2^{-1}s_3)s_1s_2^2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4$ and, expanding s_2^2, we get that this last element belongs to $R^\times(s_2^{-1}s_1^{-1})s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4+R(s_2^{-1}s_1^{-1})s_4(s_3s_2^{-1}s_3)s_1s_2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 + R(s_2^{-1}s_1^{-1})s_4(s_3s_2^{-1}s_3)s_1s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4$. Now $s_4(s_3s_2^{-1}s_3)s_1s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$ by Lemma 7.13(1) and $s_4(s_3s_2^{-1}s_3)s_1s_2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$ by Lemma 7.13(2). We are thus reduced to considering $s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 = s_4(s_3s_2^{-1}s_3s_2^{-1})s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4$ which, by Lemma 3.5, lies in

$$R^\times s_4s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 + s_4u_2u_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 + s_4u_3u_2s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4.$$

The two latter terms lie in V^+ by Lemma 7.13(4) and 7.14(2), so we are reduced to considering $s_4s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 = s_2^{-1}s_3^{-1}s_2^{-1}s_4w^-s_4w_0^2s_4$, hence to $s_4w^-s_4w_0^2s_4$.

7.9. Conclusion of the computation

By Lemma 4.9, we have $w_0^2 \in A_3^{\times} w_0^{-1} + U^+ = w_0^{-1} A_3^{\times} + U^+$, and $U^+ = A_3 w_0 + U_0 = w_0 A_3 + U_0$. We then have $s_4 w^- s_4 w_0^2 s_4 \in s_4 w^- s_4 w_0^{-1} s_4 A_3^{\times} + s_4 w^- s_4 w_0 s_4 A_3 + s_4 w^- s_4 U_0 s_4$.

On the one hand, we know that $s_4 w^- s_4 A_3 u_3 A_3 s_4 = s_4 w^- s_4 A_3 u_3 s_4 A_3 = s_4 w^- s_4 u_2 u_1 u_2 u_1 u_3 s_4 A_3 = s_4 w^- s_4 u_2 u_1 u_2 u_3 s_4 u_1 A_3 \subset V_0$ by Proposition 6.3, and that

$$\begin{aligned} s_4 w^- s_4 A_3 u_3 u_2 u_3 A_3 s_4 &= s_4 w^- s_4 A_3 u_3 u_2 u_3 s_4 A_3 = s_4 w^- s_4 u_1 u_2 u_1 (u_2 u_3 u_2 u_3) s_4 A_3 \\ &= s_4 w^- s_4 u_1 u_2 u_1 u_3 u_2 u_3 u_2 s_4 A_3 = s_4 w^- s_4 u_1 u_2 u_1 u_3 u_2 u_3 s_4 u_2 A_3 \subset V^+ \end{aligned}$$

by Lemma 7.4(4) (apply $\Phi \circ \Psi$ to the identity there). From $U_0 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 A_3$ one thus gets $s_4 w^- s_4 U_0 s_4 \subset V^+$.

On the other hand, we have $s_4 w^- s_4 w_0 s_4 \in V_0 + s_4 w^- s_4 w^+ s_4 A_3 \subset V_0$ by Lemma 6.11(5). We are thus reduced to $s_4 w^- s_4 w_0^{-1} \in s_4 w^- s_4 w^- s_4 A_3^{\times} + V_0$, which concludes the proof.

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